STUDIES IN NONLINEAR DISCRETE TIME SYSTEMS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
A. KRISHNAN

to the
DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

JANUARY, 1979

ACKNOWLEDGEMENT

It is with a deep sense of gratitude that I acknowledge the invaluable guidance received from my thesis supervisor Dr. R. Subramanian, who suggested the problem and has been a source of constant encouragement during the course of this research work.

I am greatly indebted to my brother Dr. P.A. Venkatachalam Professor, College of Engineering, Madras who is much responsib for the initiation and completion of this research program.

I find no words to express my gratitude to Dr. L.P. Singh, Dr. M. Arunachalam, Mr. Y.P. Singh, Mr. K. Gopalan and Mr. K. Natarajan for their encouragement and invaluable suggestions.

I should thank my wife Vijaya and children for their forbearance throughout the venture of this work.

I am thankful to many who have facilitated the preparation of this report, especially the assistance rendered by Mr. C.M. Abraham, who typed this manuscript.

EE-1979-D-KRI-STU

CERTIFICATE

Certified that the work presented in this thesis entitled 'STUDIES IN NONLINEAR DISCRETE TIME SYSTEMS' by Mr. A. KRISHNAN has been carried out under my own supervision and this has not been submitted elsewhere for a degree.

R Subramanian

(Dr. R. Subramanian)
Assistant Professor
Department of Electrical Engg.
I.I.T. Kanpur-16.

TABLE OF CONTENTS

LIST OF SYMBOLS

η	Fast time scale
τ	Slow time scale
k	Integer time variable
μ	Perturbation (small) constant
t	Continuous time variable
ω_n, S_n	Coefficients in fast and slow variable expression
ω_o	Natural frequency
$f(\cdot)$	Nonlinear function
$A(\tau)$, $B(\tau)$	Amplitude functions
$x(k)$	Dependent variable
x_o	Base or generating solution
x_i	Correction terms in series solution
Δ	Forward difference operator
∇	Backward difference operator
δ	Central difference operator
Δ_x, Δ_y	Forward partial difference operators
δ_x, δ_y	Central partial difference operators
$\Delta_x^2, \delta_x^2, \delta_x\delta_y$	Second order partial difference operators
I	Set of nonnegative integers
T	A general difference operator
N, M	Detuning parameters
α, β, γ	Constants

$F(k)$	Forcing function (periodic)
$g(.,.)$	Nonlinear function
R	Magnitude
θ	Phase
C_o, C_n, b_n	Coefficients in Fourier expansion
L	Integer period
P, Q	Constants
$x(o), x(1)$	Initial conditions
m	Small constant (slope)
$Z(k)$	Input to digital filters
a, b	Filter coefficients
P_L	Limit cycle oscillation of period L
x^*	Limit cycle sequence
F_1, F_2	Magnitude of input functions
ω	Input frequency
$\underline{A}$	Matrix A
$\underline{x}, \underline{y}$	Vectors x and y
$\underline{A}^T$	Transponse of $\underline{A}$
$\underline{A}^{-1}$	Inverse of $\underline{A}$
$\cdot$	d/dt

LIST OF FIGURES

LIST OF TABLES

SYNOPSIS

A. KRISHNAN
Ph.D.
Department of Electrical Engineering
Indian Institute of Technology, Kanpur.

November, 1978

STUDIES IN NONLINEAR DISCRETE TIME SYSTEMS

For nonlinear systems, described by either nonlinear differential or nonlinear difference equations there is unfortunately no general procedure for obtaining solutions in closed form. In fact, when a physical problem leads to nonlinear differential (or difference) equations, one is most often content with a qualitative description of the solution together with a numerical approximation. Most engineering situations are inherently nonlinear, and can be successfully linearised only for particular operating condition in order to apply known linear techniques. However, a considerable number of useful engineering problems can not be adequately linearised.

Both weakly and strongly nonlinear continuous time systems have been investigated by various persons. However, little work has been reported on the analysis of discrete time systems, in particular nonlinear discrete systems. Since 1960 discrete systems have gone through a rapid change in the control area and the principle of optimal control, the concept of state variables, stability studies by Liapunov's method etc. have been used for discrete systems study. The well known technique

of Krylov - Bogoliubov averaging method has also been applied to a restricted class of Nonlinear Discrete Time Systems to obtain approximate solution.

In this thesis, nonlinear difference equations are considered and a detailed analysis is carried out for both autonomous and nonautonomous situations. Nonlinear difference equations arise commonly in the analysis of various discrete time problems such as sampled data control systems, digital filters, biological oscillations etc. Analysis of nonlinear difference equations in the time domain have however been studied to a much lesser degree than their counterpart - nonlinear differential equations. The major contributions in this thesis are listed below :

1. A new method, known as discrete multiple scale perturbational technique, for analysis of nonlinear difference equations is introduced, which is sufficiently general to consider both autonomous and nonautonomous descriptions. This method is based on the introduction of two separate independent time scales - a fast time scale and a slow time scale. This technique has been used extensively to study a class of differential equations (both ordinary and partial), but there has been no instance of its application to discrete time systems except the paper by Hoppensteadt et al. dealing with a system of difference equations in matrix form, in which the discrete

time model is obtained through the well known Taylor series expansion. The proposed multiple scale perturbational technique, on the other hand is developed using the known properties of finite difference operators.

2. The discrete multiple scale perturbational technique has been applied to a class of nonlinear difference equations to obtain both qualitative and quantitative information. Linear and nonlinear difference equations under free and forced situations are analysed using the proposed technique. Super/sul harmonic oscillations under strongly forced condition are also investigated through the proposed technique.

3. The multiple time method is applicable only for a polynomia. kind of nonlinearity. The remaining part of the thesis is devoted to the analysis of nonlinear difference equations with saturation type of nonlinearities which are common in digital filters. These nonlinearities are due to adder overflow and quantization during addition or multiplication of two or more signals. The present study is limited to second order digital filters (which are in fact basic blocks for higher order systems) with overflow saturation nonlinearity. The present study has focussed on the following new investigations

(i) the existence of limit cycles with different periods for parameter values outside the stability triangle.

(ii) the location of a region outside the stability triangle in the parameter plane in which the output sequence dies down to zero monotonically.

(iii) under forced situation, conditions are derived for the existence of jump phenomenon as well as nonlinear sustained oscillations when the parameter values, the magnitude and the frequency of the input function are known beforehand.

A chapterwise summary of the work reported in this thesis is given now :

The first chapter provides an introduction to the nonlinear systems and nonlinear discrete time systems in particular. A survey of some of the earlier work pertaining to this thesis is presented. A brief description of second order digital filters, the nonlinear phenomenon introduced due to finite wordlength registers and their effect on the normal operation of the filter are discussed. The scope and objectives of the thesis are also outlined.

The second chapter is concerned with nonlinear difference equations with weak polynomial type of nonlinearity under free and weakly forced situations. The proposed discrete multiple scale perturbational scheme is applied to linear as well as a class of nonlinear difference equations and the deduced results are compared with exact solution obtained by computer simulation. For weakly nonlinear systems, a Vander pol type model is considered and its limit cycle oscillations are investigated. The stability property is studied by a variational approach. The

discrete version of the Duffing equation under free and weakly forced situations are also taken as illustrative examples and the response characteristics such as variation of amplitude with input frequency are obtained. For weakly forced and weakly damped case, the concepts like vertical and horizontal tangents in the steady state response characteristics are also studied.

Chapter three deals with the important concept of super/sub harmonic oscillations in nonlinear discrete time systems under strongly forced situation and the analysis is carried out through the multiple scale perturbational technique as well as by harmonic balancing method. A Duffing type equation is considered and the possible super/sub harmonic oscillations are obtained.

In the fourth chapter a saturation type of nonlinearity is considered which is common in digital filters. For a second order digital filter with overflow saturation nonlinearity under force free condition analytical expressions are derived to obtain different regions in the a-b parameter plane for sustenance of different periods of limit cycle oscillations. Besides the stability investigation another interesting contribution is to locate the region outside the stability triangle in the a-b parameter plane in which the output sequence decays to the zero solution monotonically for a

particular set of initial values of the filter variable. The region in the parameter plane and the initial condition loci are obtained for such a monotonic response.

The fifth chapter focusses on the investigation in digital filter under forced situation, known as 'jump phenomenon'. Jumps exist in second order digital filters, for a specific filter parameter values and the magnitude of periodic input function, whenever there is a disturbance in the filter variable. For a given input frequency, conditions are derived in terms of the filter coefficients and the magnitude of the input function for the existence of a jump or nonlinear sustained oscillation. Location of regions in the a-b parameter plane for existence of such nonlinear phenomena are also shown by graphical construction of the derived conditions. In addition to the above nonlinear phenomena in forced digital filter, the possibility of the existence of subharmonic oscillations are also investigated.

The work reported and the contributions made in this thesis are reviewed in the sixth chapter. This chapter concludes with an assessment of the scope for further research work in this area.

CHAPTER 1

INTRODUCTION

1.1. General Discussion:

The dynamic analysis of a physical system is usually a description, in mathematical form, of the physical properties of the system. This form is often called the mathematical model of the system. Most mathematical models of physical systems are integro-differential equations or difference equations, or a combination of these equations. When these equations are linear the system is said to be linear, which are in turn classified into those with constant and variable parameters. In linear systems with variable parameters, the parameters are functions of the independent variable, generally the time. These systems are also known as parametric systems. A system is nonlinear if its behaviour can not be described by linear equations. Physically this means that the parameters are functions of dependent variable and its derivatives. The most significant difference between linear and nonlinear equations, from an engineering point of view, is that linear equations obey the laws of superposition, and hence while nonlinear equations do not.

Nonlinear equations, which describe the behaviour of nonlinear systems, cannot as a rule be solved explicitly, unlike linear systems with constant coefficients. Among

nonlinear systems the separate class of self-excited systems have been the subject matter of extensive research. This class includes systems that can produce oscillations without an external force. In general nonlinear systems are an indispensable part of most engineering applications, and their study is of much importance. Various techniques for investigating the solutions are used to aid in their study. These techniques are numerous and a brief survey is given in the sections 1.2 and 1.4. The next two sections describe some salient features of nonlinear continuous time and nonlinear discrete time systems.

1.2 Nonlinear Continuous Time Systems :

Nonlinearities exist in almost all physical problems. For instance many problems in aerodynamics and hydrodynamics involve nonlinear equations [1]. In electronics a common nonlinear device is the triode oscillator. Appelton and Van der Pol [2,3] first derived a mathematical model for this circuit, described by nonlinear differential equation. In Van der Pol obtained solutions for this equation by the isocline method. In magnetic circuits, saturation of the iron core is a nonlinear phenomenon. In mechanical systems, the mass on a nonlinear spring is a nonlinear system which was studied by Duffing and others in early 1920.

In general a nonlinear system can be analysed by (a) qualitative methods and (b) quantitative methods. In

qualitative analysis, the complete information can be obtained without having to solve the given equations. The phase plane analysis and describing function techniques are some of the techniques employed. The quantitative methods are analytical in nature. The study of nonlinear systems through these methods has been a subject of much study since the time of Poincare. The analytical techniques used in nonlinear systems analysis have been developed as a result of work done in nonlinear mechanics and nonlinear circuit analysis. However, at present it appears that no generalised theory of solution can be formulated for all nonlinear systems.

The behaviour of a nonlinear system is quite different from that of a linear system. Unlike linear systems, a nonlinear system may be stable for small inputs and may become unstable if the input exceeds a certain level. The nonlinear system may respond in certain other ways which are peculiar only to this type of system. Limit cycle oscillations, super/subharmonic oscillations, frequency entrainments and jump phenomena are some of the examples for the peculiar behaviour of nonlinear systems. These are briefly explained as follows :

1.2.1 Limit cycle oscillation :

This form of oscillation in a nonlinear system is a steady state oscillation, usually periodic but not necessarily sinusoidal. In the phase plane the limit cycle oscillations

have closed curves known as periodic trajectories. Stability of such periodic trajectories are studied by considering a small variation on either sides of the closed trajectory. Periodic trajectories which are asymptotically stable from both sides are called stable limit cycles. Those which are unstable from both sides are called unstable limit cycles. The periodic trajectories which are stable from one side and unstable from the other side are known as semistable limit cycles. Some nonlinear systems may have multiple limit cycles whose trajectories are closed isolated curves.

1.2.2 Super and subharmonic oscillations :

In a linear system where input is a sinusoidal forcing function with frequency ω , the steady state output will also oscillate at frequency ω , except in the special case where the system has two or more repeated poles on the imaginary axis. In some nonlinear systems, on the other hand, for the same forcing function, the output will have a steady state oscillation that is comprised of the fundamental component at frequency ω along with components at frequency $p\omega/q$ where p and q are positive integers. When the frequency component $p\omega/q$ is predominant then the steady state oscillations are classified as follows :

(i) Super or ultraharmonic oscillations; if $q = 1$ and $p \geq 2$,

(ii) Subharmonic oscillations; if $p = 1$ and $q \geq 2$, and

(iii) Ultrasubharmonic oscillations; if p and q are relative prime integers.* (p and q are said to be relative prime integers if their greatest common divisor is unity.)

1.2.3 Frequency entrainment :

If a nonlinear system has a natural limit cycle oscillation of frequency ω_1 and if it is forced by an external input at frequency ω_2, it may be expected that components at both frequencies will appear at the output. In some cases, however if ω_2 is close to ω_1, the output component at ω_1 will vanish leaving an oscillation at ω_2 only. This phenomenon is known as frequency entrainment. The range of values for ω_2 corresponding to ω_1 for which this phenomenon occurs is called the zone of synchronization. When ω_2 is not close to ω_1, both components will generally be present. A considerable amount of literature concerning the frequency entrainment is available which includes the text book by Hayashi [4]. In this book the possible occurrence of higher/subharmonic entrainments are also discussed.

1.2.4 Jump phenomenon :

Yet another phenomenon occuring in some nonlinear systems is discontinuous jumps in output magnitude. This happens for smooth and continuous variation of either frequency or amplitude of the forcing function. For either

*where their greatest common divisor is unity.

variation, the output jumps from one value to another with widely differing magnitude. This has been analysed in great detail in many text books including Stoker [5], Cunningham [6] and Ku [7].

The existence of these and other phenomena that are not found in linear systems makes the field of nonlinear systems quite complicated for study. The theory of nonlinear systems has not been developed to the point where one can say very much about the solution of the system from the equation itself, as is possible with linear constant coefficient equations. Therefore, each nonlinear equation presents a new problem. Now a brief review of the methods available to analyse the nonlinear systems is presented below.

The methods for finding solutions for nonlinear differential equations or differential equations with varying coefficients , generally are more difficult and the solutions less satisfying on account of the degree of approximation involved, than those for linear constant coefficient differential equations. Only in few cases can exact solution in terms of known functions be found. Usually, an approximate solution is possible, and this solution may be valid with reasonable accuracy only for certain range of values of the independent variable. When the equations are nonlinear, solutions by analytical methods are possible only if the

nonlinearity is small and even then available solutions are generally approximate. The system described by a small nonlinearity is generally known as weakly nonlinear systems. Equations with a considerable degree of nonlinearity are known as strongly nonlinear systems. These systems can be solved generally by numerical or graphical methods.

Several procedureshave been discussed by Cunnigham [6], Mclachlan [8], Andronow et al [9], Gibson [10] and many others for obtaining the approximate solution for a nonlinear differential equation by analytical approach. The method of slowly varying amplitudes, power series methods and perturbational methods are commonly used to analyse such systems. The method of slowly varying amplitudes is widely used in the theory of oscillations. This method was first introduced by Van der Pol [11], who had considered a number of problems on transient phenomena in valve oscillators and other systems. This method has been further extended by Krylov and Bogoliubov [12] among others to study nonlinear systems. The technique developed by Krylov-Bogoliubov has been amplified and justified by Bogoliubov and Mitropolski [13] who have developed the powerful technique known as Krylov-Bogoliubov-Mitropolski (KBM) method for determining the approximate solutions of weakly nonlinear systems. These methods are mainly concerned with the existence of periodic solutions in weakly nonlinear systems. As mentioned earlier if the degree of nonlinearity

is sufficiently small, a first corrected solution by applying either power series or perturbation methods may give sufficient accuracy. If the degree of nonlinearity is large, it is sometimes possible to obtain better accuracy by applying these methods repeatedly. Further repeated application of these methods are possible in theory, but in practice the mathematical computations become complicated.

The perturbation technique is applicable to equations with a small parameter associated with nonlinear terms. An approximate solution is found as a power series, with terms involving the small parameter raised to successively higher powers. If the magnitude of the parameter is small enough, the first few terms of the series are sufficient to give a fairly accurate solution.

Most of the available analytical methods are limited in their application to the weakly nonlinear systems. The perturbation technique is most widely used in the field of nonlinear systems. This method accounts for the effect of nonlinearity by additive correction terms and is also relatively straightforward at least where the linear terms as well as nonlinear terms appear in the differential equations. Where the solution is oscillatory, however, secular terms with indefinitely increasing magnitude may arise, and the procedure must be modified to eliminate such terms [6]. This method may be applied successively in order to obtain

additional terms in the series solution and thereby to achieve better accuracy. Each successive application, however, generally becomes tedious.

When the solution of nonlinear equation is an oscillation having its amplitude or phase changing with time, but doing so relatively slowly, the multiple time perturbation technique is useful.

The two time scaling which is a special case of multiple time perturbation technique has been used successfully for analysis of a class of second order diffferential and partial differential equations. The formulation of two time expansion scheme was initially given by Cole and Kevorkian [14] and they obtained uniformly valid asymptotic approximations for certain nonlinear differential equations. Kevorkian [15], Cole [16], Carrier and Pearson [17] and Nayfeh [18] have discussed in great detail, the theory and the application of two variable expansion in analysing a class of nonlinear second order differential equations as well as partial differential equations. Rasmusson [19] obtained uniformly valid zeroth order approximation for the general Lienard type equation with small damping by means of multiple time scaling. The two time scaling method has been extended to analyse a set of coupled as well as higher order systems. Yen and Kronauer [20] studied the response of three oscillators

coupled by a weak nonlinearity through this approach. Recently Tiwari and Subramanian [21] have successfully applied the two time scaling technique for third order nonlinear differential equations of various classes with polynomial kind of nonlinearities. Tiwari et al [22] and Kronauer et al [23] have considered the application of two variable expansion scheme for determining super, sub and ultrasubharmonic resonances in a class of nonlinear second order differential equations with multiple and single input conditions respectively. Lick [24] and Jordan [25] have also considered the two time scaling approach to analyse the singular perturbation problems and Mathieu equations respectively. In short, the nonlinear oscillatory systems have been analysed deeply by various persons using this techniques.

During the past few years transformation techniques have also found considerable application in the study of nonlinear differential equations. Dasarathy and Srinivasan [26] used transformation methods to reduce a restricted class of nonlinear systems into equivalent linear systems. Jones and Ames [27] applied similarity methods to reduce the order of nonlinear differential equations. Musa and Srivastava [28] employed Lie's method of infinitesimal transformations to generate some general classes of second order nonlinear nonautonomous differential equations which can be reduced to first order systems. This method is most general and applicable to higher order systems too.

In addition to the above approximate methods, a method of analysing nonlinear systems known as functional analysis approach has also been established in the literature. The functional analysis viewpoint provides a broad and powerful means for treating many problems in nonlinear systems as well as in other fields like filtering and optimal control. The book by Porter [29] gives a detailed introduction to linear functional analysis with engineering motivation. A brief introduction to functional analysis approach and its application to nonlinear systems has been given by Holtzman [30]. In [30] the functional analysis approach has been successfully applied to study nonlinear systems especially periodic solution of weakly nonlinear systems and stability analysis of nonlinear systems.

1.3 Nonlinear Discrete Time Systems :

Unlike the continuous time systems, in discrete time systems the dependent variable assumes values only at discrete time instants. Difference equations bear the same relation to the discrete time systems as differential equations do to continuous time systems. As in the case of continuous time systems, the discrete time systems can also be classified as linear and nonlinear systems. The system is said to be linear if the mathematical model namely the difference equation is linear, that is the dependent variable and its differences appear only to the first power. If this

power is other than the first, or if the variables appear as product with one another, the equations are nonlinear. In general a difference equation is defined as follows. Let S denote a set of real numbers x. The set S may include all real numbers, or it may be restricted to integers, or non-negative integers, or only positive integers. Then if a function y(x) is defined over the set S, an equation relating the values of y and one or more of its differences $\Delta y, \Delta^2 y$, ... $\Delta^n y$ is called a 'Difference Equation'. The symbol Δ denotes the forward difference operator. The difference equation can also be represented in terms of central and backward difference operators [31,32,33]. A detailed description of these operators is given in Chapter two.

Difference equations also occur in combined form with differential equations, particularly in the field of stochastic processes. This form is known as difference-differential equations. In difference equations the independent variable may be discrete (usually a nonnegative interger valued variabl or continuous. In the latter case the equation is sometimes called a functional difference equation.

Difference equations not only play a role in their own right as direct mathematical models of physical phenomena but also provide the field of numerical analysis with powerful tools. By approximating differential equations with differenc equations and thereby obtaining numerical solutions, one can

study approximate numerical solutions of nonlinear differential and integral equations for which there are no known closed forms.

Difference equations arise commonly in analysis of various discrete time problems such as digital filters, sampled data control systems, biological oscillations etc. A survey paper by Jury and Tsypkin [34] indicates the spectrum of research in discrete time systems over nearly two decades since 1950. Various applications are also cited in the above reference.

In many fields of Engineering and Science the behaviour of certain systems may be modelled in terms of difference equations instead of differential equations. For instance Hsu and Yee [35] have studied the behaviour of dynamical systems governed by a simple nonlinear difference equations. Lorenz [36] in connection with meteorology and fluid flow stability problems, May [37] for population dynamics problems and Hsu and Cheng [38] for dynamical systems governed by periodic differential equations, also obtained difference equations for their models. Thus the development of theory of dynamical systems described by difference equations seems to hold promise of having applications in many areas of research.

As mentioned earlier, difference equations also exist in digital filters. Due to limitation in storing the signals

in registers, under certain circumstances a digital filter exhibits nonlinear operations. The nonlinearity introduced due to nonlinear operations is of discontinuous type and they can not be adequately represented by polynomials. Hence some special methods are to be adopted to analyse such systems. The following section is concerned with digital filters and the nonlinear phenomena associated with them.

1.3.1 Digital filters :

The nonlinearity associated with the discrete time system may be of a polynomial type or piecewise linear e.g. saturation type. The latter type of nonlinearity is common in physical systems such as relay servomechanisms and digital filters. Digital filtering is one of the important tools for modern digital signal processing and it has found many applications in an increasing number of fields in science and engineering. A digital filter may be defined [39] as a computational process or algorithm which converts one sequence of numbers representing an input signal into another sequence of numbers representing an output signal, and in which the conversion changes the character of the signal in some prescribed fashion.

Digital filters are discrete systems, with the additional property that the signals at any instant can only have a finite number of values, usually represented in binary form. This makes it possible to store the signals in

registers, necessarily having a finite length. In spite of many advantages offered by digital filters there is an inherent limitation on the accuracy of these filters due to this finite wordlength. A survey and discussion of the effect of finite wordlength on the accuracy of digital filters is given by Liu [40] and Oppenheim et al [41]. In such systems the finite wordlength available for the signals affects the perfect operation and in most cases introduces nonlinear phenomena.

The nonlinear phenomena introduced in digital filters are due to arithmetic operations, such as multiplications and additions performed on the signals. After these operations, the wordlength will, in general, have increased. Assuming a fixed point representation [42, 43] a multiplication of an N bit word with another K bit word results in a word having (N+K) bits. Similarly addition of two words of N bits will result in a word requiring (N+1) bits for its representation. This increasing word length could be handled by providing larger registers, which in most systems, would lead to very impractical size. To avoid this kind of practical difficulty, often a wordlength reduction is applied. Generally two types of wordlength reduction namely quantization and overflow are applied.

(i) Quantization :

This is obtained by either rounding a digital word if say N+K bits to N bits or supressing several of the least

significant bits (magnitude truncation). This effectively results in a nonlinear digital filter.

(ii) Overflow :

Overflow can occur after a multiplication or an addition and this affects the most significant bits. Consequently the overflow causes more severe signal distortion than the quantization.

Both of these operations result in a deviation of actual output from the desired output. Because of these nonlinear operations limit cycle oscillations can occur under zero input situation and nonlinear sustained oscillations, jump phenomena, super/subharmonic oscillations are also possible under forced input situation.

1.4 Literature Survey :

A survey of some of the earlier work in the field of nonlinear discrete time systems will now be presented. As seen in the previous section, the nonlinearity in discrete time systems can occur either in polynomial form or in a piecewise linear form. This review is mainly concerned with the past work reported in nonlinear discrete time systems with polynomial type as well as saturation type of nonlinearities.

Very little work was done on the analysis of discrete time systems prior to 1950. The early text books on Servomechanisms by Maccoll [44], James et al [45] and Oldenbourg and Sortorius [46], all have some coverage on topics of sampled data systems, mostly limited to stability analysis and derivation of transfer functions. Since 1960 the field of discrete time systems has gone through a rapid change and the principle of optimal control, the concept of state variables and stability studies by Lyapunov's method [47, 48, 49] have been used for discrete time systems study. In stability studies the methods available for continuous time systems have been modified and applied to various types of nonlinear discrete time systems; for instance Weaver and Leake [50] studied the stability and boundedness domain aspects of autonomous discrete time systems through Lyapunov's direct method. Dutchak and Sinitskiy [51] have formulated theorems to study the stability, asymptotic stability and instability of the solutions of nonautonomous nonlinear difference equations. Most of these theorems are very similar to Lyapunov's theorems for continuous case. Liapunov's direct method has also been considered by Gordon [52] to study the stability properties of perturbed difference equations and to analyse the stability aspects of discrete systems with periodic coefficients in [53]. Two recent text books Freeman [54] and

Cadzow [55] written exclusively for discrete time systems discuss at length the linear system theory aspects of discrete time systems.

As far as the analysis of difference equations are concerned, much less work has been done till recently. Milne and Thompson [31] and Jordan et al [33] among others, have considered at length the various techniques that can be used to obtain the solution of linear difference equations with constant coefficients. A detailed description of various methods to analyse the linear discrete systems is also found in Freeman [54] and Cadzow [55].

A powerful mathematical tool devised for the analysis and design of discrete time system is the Z transform. Ragazzine and Zadeh [56] formally introduced Z-transform theory and applied it to automatic control systems. Barker [57] further extended this method and suggested what is now called the modified Z-transform method. The application of the modified Z-transform method to wide class of discrete time systems are discussed in the text books by Kuo [58] and Jury [59].

Lindorff [60] has shown by introducing bilinear transformation that the Routh Hurwitz criterion and the Bode diagram can be applied to the linear discrete systems. It is also shown how the Nyquist criterion and root locus technique

are valid techniques as applied to the discrete time model. Sedlar and Bekey [61] have applied newly formulated signal flow graphs to sampled data systems. Just as the methods available for solving a linear differential equation can be extended to linear difference equations, so also the methods developed for analysing nonlinear differential equations appear extendable to nonlinear difference equations. Details of applying the methods have to be altered somewhat, but much the same general approach appears possible.

There is no general technique,however,to analyse the nonlinear difference equations as is also seen in the case of nonlinear differential equations. A considerable body of literature has been developed,however, on the subject of analysis of nonlinear differential equations. Analysis of nonlinear difference equations in the time domain has, however, been studied to a much lesser degree than their counterpart, the nonlinear differential equations.

Early work in this area can be traced to contributions by Trong [62] who presented a method for solving second order nonlinear difference equations containing small parameter. The theory of first approximation which is widely used in solving nonlinear differential equations is extended to the field of nonlinear difference equations. Huston [63] has applied the discrete analog of the Krylov-Bogoliubov's (K-B method) averaging method to the above equations. These

methods are very laborious and limited to autonomous systems. Transform techniques have been popular in control system context. Pai and Jury [64], Reddy and Jagan [65], Lavi et al [66], among others, have applied respectively, convolution, multidimensional and modified multidimensional Z-transforms for the analysis of nonlinear difference equations. Nonlinear time varying discrete systems have been considered by Jagan and Desai [67] and analysed through multidimensional Z-transforms. Here again only autonomous systems have been mentioned and these approaches are also fairly lengthy and involved. Alper [68] and Fu [69] have applied volterra series for the analysis of a class of nonlinear discrete systems and this approach is again limited to simple nonlinearities.

Aseltine and Nesbit [70] and Lindroff [71] proposed the incremental phase plane technique to analyse a nonlinear sampled data system described by difference equations. Aseltine et al [70] have explained a method to obtain the difference equation to describe a sampled data system. A coordinate transformation in the incremental phase plane is introduced [71] to identity intersample oscillations which are undetectable in the original system. Much more about the methods is to be developed for example the possibility of their use to predict the nonlinear phenomena such as limit cycle behaviour. Pai [72] has proposed a new technique to

examine the necessary conditions for the existence of certain limit cycles in nonlinear sampled data systems. This technique is based on the principle of specifying a certain repetitive output from the nonlinearity and finding conditions under which this sequence will be sustained. The analysis is based on a frequency domain approach and requires an exhaustive search of all possible periodic modes.

Second order digital filters described by nonlinear difference equations are also considered in the present study. Such systems are analysed to investigate certain nonlinear phenomena such as limit cycle oscillations, jump phenomenon and super/sub-harmonic oscillations. We now survey some of the work done in the field of nonlinear digital filters.

As mentioned earlier, because of finite wordlength available for the representation of signals, almost every digital filter is nonlinear. Effects of wordlength reduction in digital filters, such as limit cycles, overflow oscillations etc. have been known for a long time. Since the early work of Gold and Rader [73], Kaiser [74], Jackson [75] and Ebert et al [76], in which these effects were mentioned for the first time, a large number of papers have appeared dealing with this subject. Methods have been reported for the analysis of these effects, and measures for suppressing the unwanted phenomena have also been described, especially for second order wave digital filters [77, 78]. The wave filters are related to classical networks and this relation transforms

the power function of the so called reference network into a function, known as pseudo power introduced by Fettweis [77]. The associated pseudo energy function can be used as a Lyapunov function for determining the stability of wave digital filters. An introduction to wave digital filters and their properties are also found in the above references.

In the available literature on nonlinear phenomena due to wordlength reduction, a bulk of work has concentrated on limit cycles rather than on jump phenomena and sustained non-linear oscillations. Basically there are two types of limit cycles, the first is due to overflow in the adder when one's or two's complement arithmetic is utilised and the second results from the rounding or truncation of the signal coefficient product at the output of the multipliers.

Four possible overflow nonlinear characteristics are reported in a recent paper by Claasen et al [79]. The study of overflow oscillations which can occur in the response of a second order digital filter, whose linear model is asymptotically stable, was begun by Ebert et al [76] and Sandberg [80]. Ebert et al [76] have shown, for two's complement arithmetic, overflow oscillations can be sustained for appropriately chosen initial conditions, while no input signal is present. These overflow oscillations are completely eliminated when a saturation kind of overflow nonlinearity is employed. However, if quantization effects are taken into

consideration, then the zero input response of a second order digital filter having an asymptotic stable linear model, can in fact possess overflow oscillations when employing saturation arithmetic. Sandberg [80] has shown that the amplitude of overflow oscillations can be made as small as possible by making the quantization effects small, that is by using sufficiently large number of bits in the representation of the data.

Montgomery [81], Mitra et al [82], Willson [83] and Claasen et al [79] among others studied the overflow oscillations in second order digital filters with saturation arithmetic. Most of the work has been directed towards the question of existence of limit cycles, analysis [80, 84, 85], estimates on limit cycle amplitudes [86, 87, 88], stability criteria [89, 90, 91], and extension to higher order sections [92]. Parker and Hess [93] studied the limit cycle phenomena in a somewhat greater breadth and developed new amplitude bounds based on Lyapunov's method and a matrix oriented approach.

Under forced situation, depending on the initial conditions two or more entirely different steady state responses can result for the same input signal [94]. A small change in amplitude or frequency of harmonic input signal may cause a jump between two distinct stationary solutions, usually known as jump phenomenon. Kristiansson [95] and Claasen

et al [96] have studied this effect in a forced second order digital filter with saturation overflow characteristics. As in continuous time systems, the response of a filter to a harmonic input signal, contains considerable subharmonic components. This phenomenon of subharmonic oscillation has been studied by Claasen et al [96] in a recent paper. Willson [97] has proposed a general error feedback circuit to minimise the presence of nonlinear phenomena, caused by the occurrence of adder overflow in the forced response of a second order filter. In general a large number of papers have already appeared in dealing with the subject of limit cycles in digital filters and a comprehensive bibliography on the limit cycle problem is given by Kaiser [98] as well as Claasen et al [79]. On the other hand, there have been very few papers dealing with discussion of jump phenomena and super/sub-harmonic oscillations in digital filters.

1.5 Scope and Outline of the Thesis :

The review of various methods to analyse the nonlinear difference equations arising in discrete time systems, given in the previous section has made it clear that there is no general way to approach a nonlinear difference equation. A few techniques like Krylov-Bogoliubov's averaging method are available to study the nonlinear discrete systems with certain limitations. In this thesis a part of work is concerned with the application of a very recent scheme, namely multiple

time perturbational technique which has been extensively used for analysis of a class of differential as well as partial differential equations. This has been adapted to analyse nonlinear difference equations with polynomial type of nonlinearities. The main philosophy of this method is as follows. The solution x(k) is considered to be a function of two independent scales 1) a fast time η and 2) a slow time τ. The nonlinearity is essentially responsible for the modulation on a slow time scale of the solution obtained from the linear system (i.e. the nonlinearity assumed to be zero). The remaining work pertains to second order nonlinear digital filters, which are the basic building blocks for higher order filters, under free and forced situations. Limit cycle oscillations and monotonically decaying type of response, both outside the stability triangle under force free condition and the jump phenomena and subharmonic oscillations under harmonically excited situation are discussed.

The chapterwise summary of the work carried out in this thesis is given below :

In Chapter two, a multiple scale perturbational technique [99] has been developed and applied to linear as well as a class of nonlinear difference equations. For nonlinear equations the variation of output amplitude with input frequency namely the response characteristics are obtained

similar to those for continuous systems [5]. A nonlinear difference equation of Van der Pol type has been analysed and qualitative information such as limit cycle oscillations is obtained. The stability of such limit cycle oscillations is studied through a variational technique. The scheme [99] used for the above analysis is obtained through the central difference operator property. This scheme is an improved version of the one reported by Krishnan and Subramanian [100], using the forward difference operator property, to analyse a class of difference equations. Both are suitable for qualitative kind of analysis such as limit cycles, jump phenomena etc. under steady state condition, whereas the scheme reported in [100] is not satisfactory for quantitative analysis to obtain the transient solution. Whenever a transient solution of a given system is required, a modified definition to the above schemes is given, which can be applied to obtain the approximate solution closer to the exact solution. The modified scheme is formulated out of experience and the mathematics involved is less than the proposed method in [99]. In practice most of the nonlinear systems require only qualitative analysis due to complexity in the nonlinear part of the system. Examples are given to illustrate the modified technique and the results are compared with those obtained by another recent multiple scale perturbatiaval technique [101]

which is based on Taylor series expansion.

Chapter three deals with the important concept of super/sub-harmonic oscillations in nonlinear discrete time systems under strongly excited situation. The discrete multiple time perturbational technique which has been discussed in detail in Chapter two as well as the well known technique of 'Harmonic Balance' method are adapted in the present study. A Duffing kind of equation is considered and the possible super, sub and ultra sub-harmonic oscillations are investigated. The results deduced from the proposed methods are compared with the ones obtained by computer simulation of given system equation.

Chapter four concerns with study of second order digital filters with overflow saturation nonlinearity under zero input situation. The survey of the past work in this area reveals that there are no limit cycle oscillations with this type of overflow characteristics, when the quantization effects are ignored. However,zero input limit cycles do exist outside the stability triangle with saturation arithmetic. Three methods have been proposed to delineate the regions of different limit cycles in the a-b parameter plane outside the stability triangle. Another important and interesting contribution here is the identification of a region outside the stability triangle, in which the linear system is unstable and the nonlinear system is asymptotically

stable for a particular set of initial conditions. The region in the a-b plane and the set of values in the initial condition plane for which the aforesaid behaviour is possible are indicated. The system equation is simulated and the results are compared with the theoretical findings. A brief description of these two aspects namely the limit cycle oscillations and the monotonically decaying nature of the response, outside the stability triangle in the a-b parameter plane is explained in [102].

Chapter five considers forced second order nonlinear digital filters. The contributions in the field of jump phenomenon are limited and only two papers by Kristiansson [95] and Claasen et al [96] are available in literature. No mention is made anywhere in these references, of the range of the input amplitude for which there exists a jump behaviour or nonlinear sustained oscillations for a specified set of filter parameters [a,b]. A new method is proposed to find the range of input amplitude for which there exists always a jump [103], for a specified frequency and the filter coefficients. This is a simple graphical construction of derived conditions for a specified frequency and it gives regions in the a-b plane in which either jump or

nonlinear sustained oscillations are possible. The discussions on subharmonic oscillations, methods adopted to eliminate the unwanted nonlinear phenomena and the verification of results by simulation form the remaining part of this chapter.

CHAPTER 2

FREE AND WEAKLY FORCED NONLINEAR DIFFERENCE EQUATIONS

2.1 Introduction:

The present chapter is concerned with nonlinear difference equations with a polynomial kind of nonlinearity. With the advent of high speed digital computers, discrete time systems described by difference equations have been an important and practical model for the scientists and engineers, especially the control engineers. As a result, the theory of difference equations has been rapidly developing in the literature. Much of this research has been devoted to the development of results which parallel those for differential equations.

In the theory of oscillations, the method of linearisation has acquired a special significance and is widely used to analyse nonlinear oscillatory systems. This subject is covered in many text books including Krylov-Bogoliubov [12] who formulated the powerful scheme presently known as Krylov-Bogoliubov (K-B method) averaging technique to analyse a large class of nonlinear continuous systems. An amplified version of this method known as Krylov-Bogoliubov - Mitropolski (KBM) method dealing with the asymptotic theory of oscillatory behaviour of a class of nonlinear systems is due to Bogoliubov

and Mitropolski [13]. The K-B averaging method has also been adopted by Huston [63] and Trong [62] to analyse a class of nonlinear discrete time systems. The application of K-B averaging technique to nonlinear difference equations has however been limited to autonomous cases and the mathematical computation involved is lengthy. Transform methods [64-67] have been used extensively to solve linear and nonlinear difference equations. These methods, again are restricted to systems without forcing.

A recent method, namely the multiple time perturbation technique, introduced by Cole and Kevorkian [14], has been applied extensively to study a class of second order differential and partial differential equations by a number of workers including Cole [16] and Nayfeh [18]. The same scheme has been extended by Tiwari and Subramanian [21] to study the response characteristics in weakly third order nonlinear differential equations under the free and forced situations. As far as analysis of nonlinear difference equations are concerned, this multiple time scaling approach does not seem to have been attempted so far except for a recent work by Hoppensteadt and Miranker [101] who considered a general nth order discrete time system described in matrix form. In [101] the discrete time model has been obtained through a Taylor series expansion. A new technique is adopted in this thesis to develop the two discrete time scaling model through

the known properties of finite difference operators. A brief review of the properties of the finite difference operators is given in the next section. The proposed two time scaling approach is quite general and can be applied to study linear as well as a class of nonlinear difference equations with and without a forcing function. To illustrate the idea, a Van der Pol type of problem is considered and limit cycle oscillations under steady state conditions are obtained. The stability aspects of such limit cycle oscillations are also analysed through a variational approach. A discrete version of Duffing equation is then studied fully under both free and weakly forced solutions. The discrete model of continuous time Duffing equation is obtained by exploiting some properties of the central difference operator. The variation of steady state amplitude with input frequency for weakly forced situation, known as response curves are derived and analysed.

The proposed discrete two time expansion method using finite difference operator properties is more suitable for qualitative analysis of nonlinear phenomena, namely description of the steady state solution like limit cycle oscillations, response characteristic curves etc. On the other hand, the transient solution results deviate somewhat from the exact solution. To obtain more accurate result closer to the exact solution, the initially proposed scheme of two

time scaling is redefined and the results obtained using the second form are compared with those obtained using the method proposed in [101], as well as the exact solution obtained by computer simulation of the given system equation. The modified scheme proves to be a better approach for discrete problem analysis under certain conditions which are explained later on in the analysis. The results obtained are verified by computer simulation.

2.2 Multiple Time Perturbation Method in Continuous Time Syste

Before proceeding with the basic analysis it is useful to review the method of the multiple variable expansion procedure employed to analyse a class of second order nonlinear differential equations. The two time scaling method, a special case of multiple scale perturbational technique has been used to analyse a variety of weakly nonlinear differential equations (sometimes known as small perturbation problems). The two time scaling technique assumes variation of the dependent variable x interms of two independent time scales, a fast time η and a slow time τ. The motivation for introducing the two independent variables has been explained in the introductory chapter. In general both these independent time scales are assumed to be a function of a small parameter μ, usually known as perturbation parameter and are expressed as [20,104]

$$\begin{aligned} \eta &= t(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots.) \quad \text{the fast time scale.} \\ \tau &= t(\mu + S_1\mu^2 + S_2\mu^3 + \ldots.) \quad \text{the slow time scale} \end{aligned} \tag{2.1}$$

where the constants ω_n and S_n are to be determined so that the asymptotic expansion is uniformly valid. The constants S_n are included in the definition of slow time scale to allow for general dependence of nonlinear function on μ [104].

Cole [16] and Nayfeh [18] have considered the definition of the slow variable in a slightly different manner by assuming the constants $S_n = 0$. However, as far as the nonlinear systems analysis is concerned, second approximation is possible in theory and impossible in certain situations. Under such situations the definition of independent scales can be simplified by assuming $\omega_n = s_n = 0$, that is

$$\begin{aligned} \eta &= t \\ \tau &= \mu t \end{aligned} \tag{2.2}$$

Generally the application of multiple scale procedure is limited to weakly nonlinear systems and the contributions of terms μ^2, μ^3 are less significant and hence for the first approximation the independent scales defined in eqn. (2.2) can be used for analysis.

Despite the fact that solutions generated by the two time scaling method appear to satisfy the equations uniformly for all time, in general it is known that the approximate

solution is a valid approximation only for times $O(1/\mu)$ as μ approaches zero. The relation among the dependent variable x, the perturbation parameter μ and the two time scales is given as an asymptotic series as follows.

$$x(t) = x(\eta,\tau) = x_0(\eta,\tau) + \mu x_1(\eta,\tau) + \mu^2 x_2(\eta,\tau) + \ldots \quad (2.3)$$

where μ is small, that is $0 < \mu << 1.0$. In eqn. (2.3) $x_0(\eta,\tau)$ is known as 'generating' or 'base' solution and $x_1(\eta,\tau)$, $x_2(\eta,\tau)$, etc. are the first, second, etc. correction terms. It is to be noted that for weakly nonlinear systems, the terms upto first correction term in the series solution gives fairly accurate approximation for all time. The accuracy of the solution can be further improved with the addition of more correction terms. For practical applications terms upto $x_1(\eta,\tau)$ are sufficient.

The basic idea behind the two time expansion technique may be illustrated by considering a general second order differential equation of the term

$$\ddot{x} + \omega_0^2 x + \mu f(x,\dot{x}) = 0 \quad (2.4)$$

where ω_0 is the natural frequency, μ the perturbation parameter and f is a nonlinear function of indicated variables. It is assumed that the function f is differentiable with respect to its arguments. '$\cdot$' denotes the derivative with respect to time t.

The solution $x(t)$ to the eqn. (2.4) is considered as a function of two time scales and is given by

$$x(t) = x(\eta,\tau) \tag{2.5}$$

which in turn can be expressed in an asymptotic series form as

$$x(\eta,\tau) = x_0(\eta,\tau) + \mu x_1(\eta,\tau) + \mu^2 x_2(\eta,\tau) + \ldots \tag{2.6}$$

Now the dependent variable x is a function of two independent variables η and τ, the time derivatives $\dot{x}$ and $\ddot{x}$ can be expressed as partial derivatives with respect to the two time scales as follows :

$$\dot{x} = \frac{dx(t)}{dt} = \frac{dx(\eta,\tau)}{dt} = (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)\frac{\partial x(\eta,\tau)}{\partial \eta}$$

$$+ (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)\frac{\partial x(\eta,\tau)}{\partial \tau}$$

$$\ddot{x} = \frac{d^2x(t)}{dt^2} = \frac{d^2x(\eta,\tau)}{dt^2} = (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)^2\frac{\partial^2 x(\eta,\tau)}{\partial \eta^2}$$

$$+ 2(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots.)$$

$$(\mu + s_1\mu^2 + s_2\mu^3 + \ldots)\frac{\partial^2 x(\eta,\tau)}{\partial\eta\,\partial\tau} + (\mu + s_1\mu^2 + s_2\mu^3 + \ldots$$

$$\frac{\partial^2 x(\eta,\tau)}{\partial\tau^2}$$

which can be rewritten as

$$\dot{x} = \frac{\partial x(\eta,\tau)}{\partial \eta} + \mu \frac{\partial x(\eta,\tau)}{\partial \tau} + \mu^2 [\omega_1 \frac{\partial x(\eta,\tau)}{\partial \eta} + s_1 \frac{\partial x(\eta,\tau)}{\partial \tau}] + O(\mu^3)$$

$$\ddot{x} = \frac{\partial^2 x(\eta,\tau)}{\partial \eta^2} + 2\mu \frac{\partial^2 x(\eta,\tau)}{\partial \eta \partial \tau} + \mu^2 [2\omega_1 \frac{\partial^2 x(\eta,\tau)}{\partial \eta^2}$$

$$+ 2s_1 \frac{\partial^2 x(\eta,\tau)}{\partial \eta \partial \tau} + \frac{\partial^2 x(\eta,\tau)}{\partial \tau^2}] + O(\mu^3) \qquad (2.7)$$

The nonlinear function f can be expressed as follows :

$$f(x,\dot{x}) = f[x(\eta,\tau), \frac{\partial x(\eta,\tau)}{\partial \eta} + \mu \frac{\partial x(\eta,\tau)}{\partial \tau}$$

$$+ \mu^2 (\omega_1 \frac{\partial x(\eta,\tau)}{\partial \eta} + s_1 \frac{\partial x(\eta,\tau)}{\partial \tau}]$$

$$= f[x_o(\eta,\tau) + \mu x_1(\overset{\eta}{\cancel{\phi}},\tau) + \mu^2 x_2(\eta,\tau) + \ldots., \frac{\partial x_o(\eta,\tau)}{\partial \eta}$$

$$+ \mu \frac{\partial x_1(\eta,\tau)}{\partial \eta} + \ldots. + \mu \frac{\partial x_o(\eta,\tau)}{\partial \tau} + \mu^2 \frac{\partial x_1(\eta,\tau)}{\partial \tau} + \ldots$$

$$+ \mu^2 (\quad)]$$

$$= f\{ x_o(\eta,\tau) + \mu x_1(\eta,\tau) + \mu^2 x_2(\eta,\tau) + \ldots , \frac{\partial x_o(\eta,\tau)}{\partial \eta}$$

$$+ \mu [\frac{\partial x_1(\eta,\tau)}{\partial \eta} + \frac{\partial x_o(\eta,\tau)}{\partial \tau}] + \mu^2 [\quad] \ldots.\}.$$

Expanding the function f on the right hand side using Taylor series, we obtain

$$f(x,\dot{x}) = f(x_o, \frac{\partial x_o}{\partial \eta}) + \mu [x_1 \frac{\partial f}{\partial x}\Big|_{\substack{x=x_o \\ \dot{x}=\frac{\partial x_o}{\partial \eta}}} + (\frac{\partial x_1}{\partial \eta} + \frac{\partial x_o}{\partial \tau}) \frac{\partial f}{\partial \dot{x}}\Big|_{\substack{x=x_o \\ \dot{x}=\frac{\partial x_o}{\partial \eta}}}]$$

$$+ \mu^2 [\quad] .$$

Then the given eqn. (2.4) becomes

$$\frac{\partial^2 x(\eta,\tau)}{\partial \eta^2} + \omega_o^2 x(\eta,\tau) + 2\mu \frac{\partial^2 x(\eta,\tau)}{\partial \eta \partial \tau} + \mu^2 [2\omega_1 \frac{\partial^2 x(\eta,\tau)}{\partial \eta^2}$$

$$+ 2s_1 \frac{\partial^2 x(\eta,\tau)}{\partial \eta \partial \tau} + \frac{\partial^2 x(\eta,\tau)}{\partial \tau^2}] + \mu \{ f(x_o, \frac{\partial x_o}{\partial \eta})$$

$$+ \mu [x_1 \frac{\partial f}{\partial x}\Big|_{\substack{x=x_o \\ \dot{x}=\frac{\partial x_o}{\partial \eta}}} + (\frac{\partial x_1}{\partial \eta} + \frac{\partial x_o}{\partial \tau})\Big|_{\substack{x=x_o \\ \dot{x}=\frac{\partial x_o}{\partial \eta}}}] \}$$

combining the eqn. (2.6) with the above expression and collecting coefficients of μ^o, μ^1, μ^2, ... terms and equating them to zero separately, the following set of linear partial differential equations are obtained

$$\frac{\partial^2 x_o(\eta,\tau)}{\partial \eta^2} + \omega_o^2 x_o(\eta,\tau) = 0 \tag{2.8}$$

$$\frac{\partial^2 x_1}{\partial \eta^2} + \omega_o^2 x_1(\eta,\tau) = -2 \frac{\partial^2 x_o(\eta,\tau)}{\partial \eta \partial \tau} - f(x_o, \frac{\partial x_o}{\partial \eta}) \tag{2.9}$$

$$\frac{\partial^2 x_2(\eta,\tau)}{\partial \eta^2} + \omega_o^2 x_2(\eta,\tau) = -2 \frac{\partial^2 x_1(\eta,\tau)}{\partial \eta \partial \tau}$$

$$- [2\omega_1 \frac{\partial^2 x_o(\eta,\tau)}{\partial \eta^2} + 2s_1 \frac{\partial^2 x_o(\eta,\tau)}{\partial \eta \partial \tau} + \frac{\partial^2 x_o(\eta,\tau)}{\partial \tau^2}]$$

$$- [x_1 \frac{\partial f}{\partial x}\Big|_{\substack{x=x_o \\ \dot{x}=\frac{\partial x_o}{\partial \eta}}} + (\frac{\partial x_1}{\partial \eta} + \frac{\partial x_o}{\partial \tau}) \frac{\partial f}{\partial \dot{x}}\Big|_{\substack{x=x_o \\ \dot{x}=\frac{\partial x_o}{\partial \eta}}}] \tag{2.10}$$

The solution to (2.8) is the base solution and is expressed as

$$x_0(\eta,\tau) = A(\tau)\cos\omega_0\eta + B(\tau)\sin\omega_0\eta \qquad (2.11)$$

where $A(\tau)$ and $B(\tau)$ are the slowly varying amplitude functions of slow time variable τ and can be evaluated by considering the first correction equation given in (2.9).

To obtain the first correction term, the generating solution is now substituted in the right hand side of the eqn (2.9) and supression of secular terms by collecting coeffi-cients of $\cos\omega_0\eta$ and $\sin\omega_0\eta$ terms and equating them separately to zero result in coupled first order differential equation in $A(\tau)$ and $B(\tau)$. The solution to these coupled equation gives the explicit expressions for the amplitude functions $A(\tau)$ and $B(\tau)$. It is to be noted here that the explicit solutions for the amplitude functions $A(\tau)$ and $B(\tau)$ are possible for linear and for some simple nonlinear cases. However, numerical methods can be adapted to extract the information regarding the amplitude functions. Then the eqn. (2.9) can be solved for $x_1(\eta,\tau)$ with nonsecular terms on the right hand side. Then the approximate solution upto first correction term, that is upto order μ, is given by

$$x(k) = x(\eta,\tau) \simeq x_0(\eta,\tau) + \mu\, x_1(\eta,\tau)\,. \qquad (2.12)$$

In a similar way the second correction term, namely $x_2(\eta,\tau)$ can be added with the first corrected solution given in eqn. (2.12) to improve the accuracy upto order μ^2. This kind of repeated addition of correction terms are possible in theory and generally impossible in practice.

An excellent account of this method of study, both in depth and breadth, can be found in Cole [16], Nayfeh [18] and Reiss [105].

Inspite of many advantages, the multiple time expansion procedure suffers due to nonavailability of general procedure to express the two independent time scales in terms of the perturbation parameter μ. However, in a very recent paper Lovine and Obi [106] have analysed nonlinear differential equations adapting the two time scales generated from the given differential equation.

Preliminaries in the finite difference schemes which are useful to derive the multiple discrete time perturbation technique are now given.

2.3 Finite Difference Schemes :

Before proceeding with the derivation of discrete multiple time perturbational technique, it is useful to review some standard results in finite difference schemes. These results will be used further on in the analysis. For a given real

function of real variable $f(x)$, following first and second difference operators are defined [33].

(i) forward difference operator (Δ)

$$\Delta f(x) = [f(x+h) - f(x)]$$

$$\Delta^2 f(x) = [f(x+2h) - 2f(x+h) + f(x)]$$

the step size or increment h can be normalised to unity without loss of generality and the above equations take the form

$$\Delta f(x) = f(x+1) - f(x)$$

$$\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x) \tag{2.13}$$

(ii) central difference operator (δ)

$$\delta f(x) = f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2})$$

$$\delta^2 f(x) = f(x+1) - 2f(x) + f(x-1) \tag{2.14}$$

the first central difference operator δ may also be defined as follows :

$$\delta f(x) = [f(x+1) - f(x-1)]/2 \tag{2.15}$$

(iii) backward difference operator (∇)

$$\nabla f(x) = f(x) - f(x+1)$$

$$\nabla^2 f(x) = f(x) - 2f(x+1) + f(x+2) \tag{2.16}$$

Then generalising to a function of two variables x and y the partial difference operators are defined as follows :

(iv) Forward partial difference operators :

$$\Delta_x f(x,y) = f(x+1,y) - f(x,y)$$

$$\Delta_y f(x,y) = f(x,y+1) - f(x,y)$$

$$\Delta_x^2 f(x,y) = f(x+2,y) - 2f(x+1,y) + f(x,y)$$

$$\Delta_y^2 f(x,y) = f(x,y+2) - 2f(x,y+1) + f(x,y)$$

$$\Delta_x \Delta_y f(x,y) = f(x+1, y+1) - f(x+1,y) - f(x,y+1) + f(x,y) \qquad (2.17)$$

(v) the central partial difference operators :

$$\delta_x f(x,y) = f(x+\tfrac{1}{2},y) - f(x-\tfrac{1}{2},y)$$

$$\delta_y f(x,y) = f(x,y+\tfrac{1}{2}) - f(x,y-\tfrac{1}{2})$$

$$\delta_x^2 f(x,y) = f(x+1,y) - 2f(x,y) + f(x-1,y)$$

$$\delta_y^2 f(x,y) = f(x,y+1) - 2f(x,y) + f(x,y-1)$$

$$\delta_x \delta_y f(x,y) = f(x+\tfrac{1}{2},y+\tfrac{1}{2}) - f(x+\tfrac{1}{2},y-\tfrac{1}{2}) - f(x-\tfrac{1}{2}, y+\tfrac{1}{2}) + f(x-\tfrac{1}{2}, y-\tfrac{1}{2}) . \qquad (2.18)$$

In the following section, the development of discrete two time scaling technique using the above finite difference operator properties will be illustrated.

2.4 Discrete two Time Scales Expansion :

Knowing the properties of various finite difference operators discussed in the previous section, a two time scaling model can easily be obtained for discrete time

systems. The aim is to derive a stable (stable in the sense that the solution obtained by the derived technique is same as that of exact solution qualitatively) discrete multiple time perturbation technique, which when applied to difference equations would give approximate solution for all discrete time instants. Roscue [107] formulated a new stable difference scheme for differential equations which have exact solutions in certain simple situations. Numerical solution of partial differential equations has been studied by Smith [108] by replacing the derivatives with valid difference schemes. Based on these schemes a valid discrete time scaling scheme, analog to two variable expansion method in continuous time systems, is derived as follows.

Let T be a general difference operator representing a forward, a backward or a central difference operator. Let I denote the set of nonnegative integers. The two independent time variables are defined as follows :

$$\begin{aligned} \eta &= k(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots) && \text{fast time scale} \\ \tau &= k(1 + s_1\mu^2 + s_2\mu^3 + \ldots) && \text{slow time scale} \end{aligned} \qquad (2.19)$$

where $k \epsilon I$, ω_n and s_n are constants to be determined so that the asymptotic expansion is valid uniformly. Note that the general dependence of nonlinear function upon μ demanding the inclusion of the constants in the definition of τ [104] is

not true in the discrete time definition. This will be illustrated later on in the analysis.

Now the given dependent variable $x(k)$ is a function of two independent scales η and τ and is expressed as

$$x(k) = x(\eta,\tau) \tag{2.20}$$

As mentioned earlier T is a general difference operator, which operates on the associated function with respect to k, operating the eqn. (2.20) with T

$$Tx(k) = Tx(\eta,\tau).$$

In the above equation, the arguments of the variable x are η,τ which are in turn functions of k. Under this situation the right hand side operation of the above equation is defined as follows

$$Tx(k) = T_{\eta}\, x(\eta,\tau)\; T\eta + T_{\tau}\, x(\eta,\tau)\; T\tau \tag{2.21}$$

where T_{η} and T_{τ} are the general partial difference operators.

It is to be noted that irrespective of the kind of operation that T executes, $T\eta = (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots.)$ and $T\tau = (\mu + s_1\mu^2 + s_2\mu^3 + \ldots..)$. This invariant property can easily be checked as follows for various finite difference operators.

(i) $T = \Delta$, the forward difference operator.

$$T\eta = \Delta\, \eta = \Delta\,(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)\, k$$
$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots.)\,\Delta\, k = (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)$$

$$T\tau = \Delta\ \tau = \Delta\ (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)k$$

$$= (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)\Delta\ k = (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)$$

(ii) $T = \nabla$, the backward difference operator

$$T\eta = \nabla\eta = (1 + \omega_1\mu^2 + \ _2\mu^3 + \ldots)k$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots.)\ \nabla k$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots.)$$

$$T\tau = \nabla\tau = \nabla(\mu + s_1\mu^2 + s_2\mu^3 + \ldots)k$$

$$= (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)\nabla k = (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)$$

(iii) $T = \delta$, the central difference operator

$$T\eta = \delta\eta = \delta(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)k$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)\delta k$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)(k + \tfrac{1}{2} - k + \tfrac{1}{2})$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)$$

$$T\tau = \delta\tau = \delta(\mu + s_1\mu^2 + s_2\mu^3 + \ldots)k = (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)$$

$$= (\mu + s_1\mu^2 + s_2\mu^3 + \ldots.)$$

(iv) $T = \delta$, the central difference operator defined as in eqn. (2.15).

$$T\eta = \delta\eta = \delta(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots\ldots)k$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)\delta k$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)\ \frac{(k+1-k+1)}{2}$$

$$= (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)$$

$$T\tau = \delta\tau = \delta(\mu + s_1\mu^2 + s_2\mu^3 + \ldots)k$$

$$= (\mu + s_1\mu^2 + s_2\mu^3 + \ldots\ldots)\delta k$$

$$= (\mu + s_1\mu^2 + s_2\mu^3 + \ldots\ldots)\,.$$

Based on the above invariant property eqn. (2.21) can be rewritten as follows

$$Tx(k) = (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)\ T_\eta\ x(\eta,\tau) + + (\mu + s_1\mu^2 + s_2\mu^3 + \ldots\ldots)\ T_\tau\ x(\eta,\tau)\,. \qquad (2.22)$$

Operating on the above equation again by T, we get

$$T^2x(k) = (1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots)^2\ T_\eta^2\ x(\eta,\tau) + 2(1 + \omega_1\mu^2 + \omega_2\mu^3 + \ldots\ldots)\,x$$

$$(\mu + s_1\mu^2 + s_2\mu^3 + \ldots)\ T_\eta T_\tau\ x(\eta,\tau) + (\mu + s_1\mu^2 + s_2\mu^3 + \ldots)^2\ T_\tau^2\ x(\eta,\tau)\,. \qquad (2.23)$$

Eqns. (2.22) and (2.23) can conveniently be written as follows

$$Tx(k) = T_\eta\ x(\eta,\tau) + \mu T_\tau\ x(\eta,\tau) + + \mu^2[\ \omega_1\ T_\eta\ x(\eta,\tau) + s_1\ T_\tau\ x(\eta,\tau)] + O(\mu^3)\,. \qquad (2.24)$$

$$T^2x(k) = T_\eta^2\, x(\eta,\tau) + 2\mu T_\eta T_\tau\, x(\eta,\tau)$$
$$+ \mu^2[2\,\omega_1 T_\eta^2\, x(\eta,\tau) + 2s_1 T_\eta T_\tau\, x(\eta,\tau) + T_\tau^2\, x(\eta,\tau)] + O(\mu^3)\,. \qquad (2.25)$$

Equations (2.24) and (2.25) give the general discrete two time expansion scheme. A stable difference scheme can now be obtained replacing T by Δ in eqn. (2.24) and T by δ in eqn. (2.25). It is to be noted that the replacement of T by δ in both the equations gives an improved stable scheme, but this scheme is not accessible for analysis of the resulting difference equation, both by analytical means as well as by computer due to the noninteger arguments appearing as subscripted variables. Thus, the scheme used in this thesis for the analysis of discrete time system is given below :

$$\Delta x(k) = \Delta_\eta\, x(\eta,\tau) + \mu\,\Delta_\tau\, x(\eta,\tau) + \mu^2[\,\omega_1 \Delta_\eta\, x(\eta,\tau)$$
$$+ s_1 \Delta_\tau\, x(\eta,\tau)] + O(\mu^3)$$

$$\delta^2\, x(k) = \delta_\eta^2\, x(\eta,\tau) + 2\mu\,\delta_\eta\,\delta_\tau\, x(\eta,\tau) + \mu^2[2\,\omega_1 \delta_\eta^2\, x(\eta,\tau)$$
$$+ 2s_1\,\delta_\eta\,\delta_\tau\, x(\eta,\tau) + \delta_\tau^2\, x(\eta,\tau)] + O(\mu^3) \qquad (2.26)$$

where

Δ is the forward difference operator, defined as

$$\Delta\, x(k) = x(k+1) - x(k) \quad \text{and}$$

δ is the central difference operator, defined as

$$\delta\, x(k) = x(k+\tfrac{1}{2}) - x(k-\tfrac{1}{2})$$

and Δ_η Δ_τ, δ_η and δ_τ are the partial difference operators.

In general, most of the discrete time systems are described by difference equations of the following form

$$x(k+1) + \alpha x(k) + \beta x(k-1) + g[x(k),\, x\,(k-1)] = 0$$

rather than using difference operators, for instance

$$\delta^2\, x(k) + \alpha\delta x(k) + \beta x(k) + g[x(k),\, \delta x(k)] = 0$$

and hence the expressions given in eqn. (2.26) are expanded and rewritten as follows :

$$\begin{aligned} x(k+1) = {} & x(\eta+1,\tau) + \mu[x(\eta,\tau+1) - x(\eta,\tau)] \\ & + \mu^2[\,\omega_1 x(\eta+1,\tau) + s_1 x(\eta,\tau+1) \\ & - (\omega_1 + s_1)\, x(\eta,\tau)] + 0(\mu^3) \end{aligned} \tag{2.27}$$

$$\begin{aligned} x(k+1) + x(k-1) = {} & x(\eta+1,\tau) + x(\eta-1,\tau) + \mu[2x(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) \\ & - 2x(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - 2x(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + 2x(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})] \\ & + \mu^2\{\, 2\,\omega_1 x(\eta+1,\tau) - (4\,\omega_1+2)\, x(\eta,\tau) + 2\,\omega_1\, x(\eta-1,\tau) \\ & \qquad + x(\eta,\tau+1) + x(\eta,\tau-1) + 2s_1[x(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) \\ & - x(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - x(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + x(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})]\} + 0(\mu^3) \end{aligned} \tag{2.28}$$

From the above two equations

$$x(k-1) = x(\eta-1,\tau) + \mu[2x(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - 2x(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - 2x(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) +$$
$$2x(n-\tfrac{1}{2},\tau-\tfrac{1}{2}) - x(\eta,\tau+1) + x(\eta,\tau)] +$$
$$\mu^2\{\omega_1 x(\eta+1,\tau) - (3\omega_1 - s_1+2)\, x(\eta,\tau) +$$
$$2\omega_1 x(\eta-1,\tau) + (1-s_1)\, x(\eta,\tau+1) + x(\eta,\tau-1) +$$
$$2s_1[x(n+\tfrac{1}{2},\tau+\tfrac{1}{2}) - x(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - x(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) +$$
$$x(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})]\} + O(\mu^3) . \qquad (2.29)$$

The equations (2.27) - (2.29) are used to transform the given difference equation into two discrete time model. The system equation and its analysis using the above equations are described in the following sections.

Remark :

The operator T in eqn. (2.24) can also be replaced by δ, defined in eqn. (2.15) as

$$\delta\, x(k) = \frac{x(k+1) - x(k-1)}{2} \quad (*)$$

but the difficulty here is the order of the system is increased by one. So, whenever δ is appearing by itself in any equation, the replacement is by the forward difference operator Δ , namely

$$\delta x(k) = \Delta\, x(k) = x(k+1) - x(k) .$$

On the other hand when δ is appearing along with higher order difference operators namely $\delta^2, \delta^3, \ldots$ etc., the replacement for δ is as defined in (*). However, the replacement of δ with usual central difference operation namely

$$\delta\, x(k) = x(k+\tfrac{1}{2}) - x(k-\tfrac{1}{2})$$

is possible when explicit expression for $x(k)$ is known.

2.5 System Model :

A class of nonlinear discrete time systems considered in this chapter for analysis is a second order difference equation of the form

$$x(k+1) + \alpha x(k) + \beta x(k-1) + \mu\gamma g[x(k),\ x(k-1)] = \mu\, F(k) \tag{2.30}$$

where $k \in I$, is the indexing discrete time parameter, α, β, γ are constants and μ is a small positive constant. $F(k)$ is the external input, usually a periodic function. $g(\cdot,\cdot)$ is a general nonlinear function with indicated variables.

For $\mu = 0$, eqn. (2.30) reduces to a linear difference equation and is given by

$$x(k+1) + \alpha x(k) + \beta x(k-1) = 0\ . \tag{2.31}$$

It is assumed that α, β are so chosen so as to guarantee a bounded response. The different type of solutions for this

linear case are given in Appendix A. It is to be noted that for $[\alpha, \beta]$ lying anywhere within the stability triangle ACG shown in Fig. 2.1, a bounded response (that is decays to zero solution) is obtained. For $[\alpha, \beta]$ lying outside the stability triangle including the points A and C ~~are~~ on the triangle ACG, an unbounded response is obtained. On the other hand an oscillatory solution is possible on the line AC and at point G (except at points A and C), and steady state oscillatory solutions are possible on the boundary lines AG and CG. With this knowledge of the properties of the stability triangle, the analysis of the nonlinear system (for $\mu \neq 0$) will be provided in the next section.

2.6 Analysis :

With the complete knowledge about the solution of the linear difference equation (2.31), the approximate solution of nonlinear equation (2.30) for $\mu \neq 0$, will now be obtained.

Before proceeding with the detailed analysis, it is convenient to make a few comments regarding the nature of the solution for the linear system (for $\mu = 0$). An observation of the types of bounded response reveals that for $[\alpha, \beta]$ lying on the boundary of the stability triangle ACG (points A, C excluded), a steady state periodic solution is obtained. For $[\alpha, \beta]$ lying in the in~~f~~terior of ACG it is convenient to define a detuning parameter N such that for the force tree

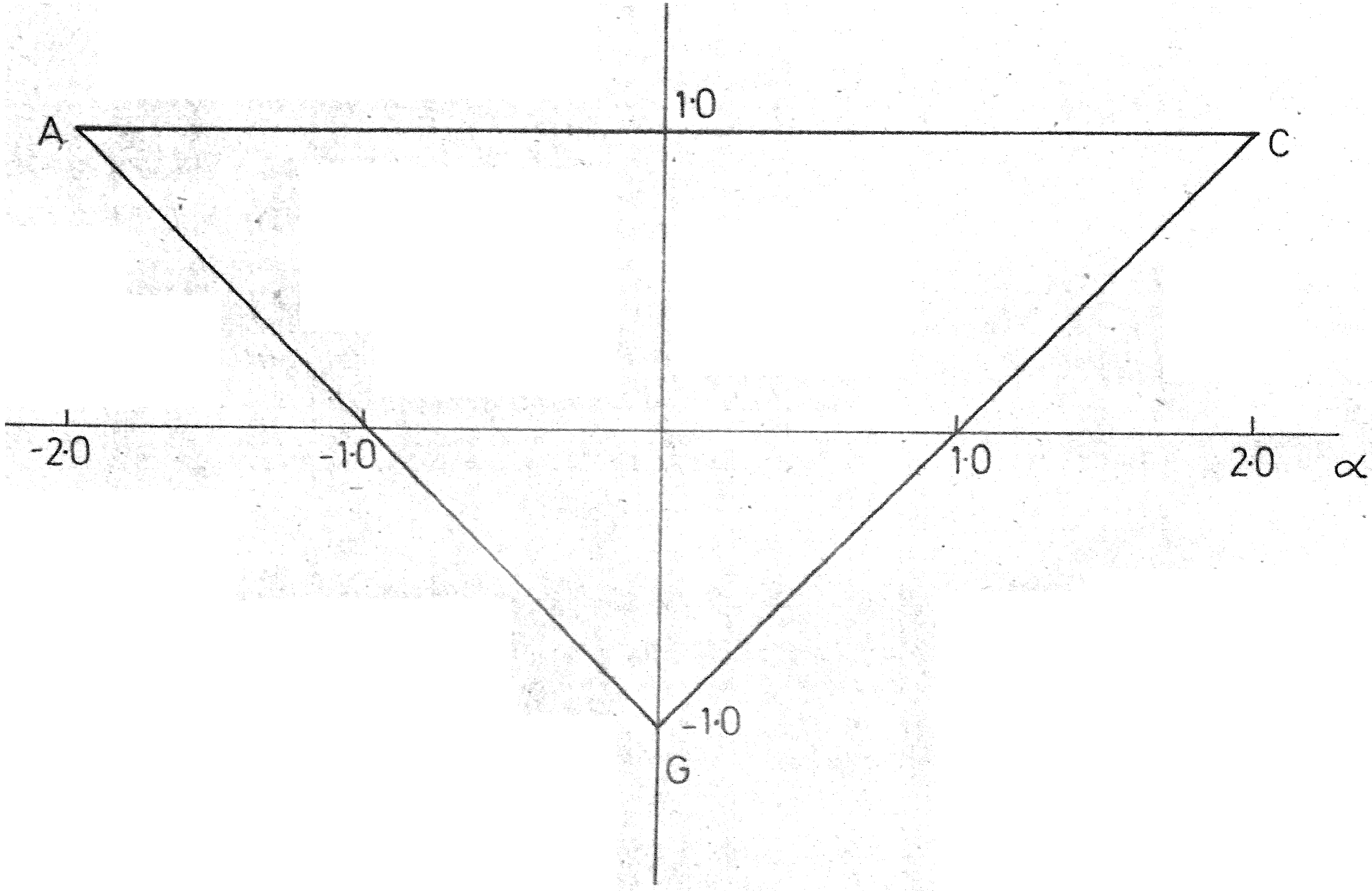


FIG.2·1 STABILITY TRIANGLE

system the discription takes the term

$$x(k+1) + \alpha x(k) + x(k-1) + \mu N x(k-1) + \mu\gamma g[x(k), x(k-1)] = 0 \qquad (2.32)$$

where N is the detuning parameter defined as

$$\mu N \overset{\Delta}{=} (\beta - 1) .$$

The advantage gained by such a transposition of terms is the generation of a 'base' solution as a periodic one (corresponding μ^0 term in (2.32)). For detuning parameter N of the order of unity or less, a good approximation to the true solution is then easily obtained as pointed out later in the analysis. Again for the forced situation, that is for $F(k) \neq 0$, it is likewise convenient to define amother detuning factor that takes into account the deviation between input frequency ω (F(k) assumed periodic) and linear system frequency. It can be expected that for small values of this detuning parameter a strong (large amplitude) response will then be obtained.

Considering $F(k) = F\cos\omega k$ and adding the term $2\cos\omega\, x(k)$ on both sides of eqn. (2.30) and rearranging, we get

$$x(k+1) - 2\cos\omega\, x(k) + x(k-1) + \mu N x(k-1) + \mu\gamma g[x(k), x(k-1)] + \mu M x(k) = \mu F \cos\omega k \qquad (2.33)$$

from eqns. (2.34) and (2.35), the following sets of linear difference equations are now obtained by considering the terms with coefficients μ^0, μ^1, μ^2, ... and equating them individually t zero.

μ^0 terms :

$$x_0(\eta+1,\tau) - 2\cos\omega\, x_0(\eta,\tau) + x_0(\eta-1,\tau) = 0 \qquad (2.36)$$

μ^1 terms :

$$\begin{aligned} x_1(\eta+1,\tau) - 2\cos\omega\, x_1(\eta,\tau) + x_1(\eta-1,\tau) = \\ -\{\, 2x_0(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - 2x_0(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - \\ 2x_0(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + 2x_0(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2}) + Nx_0(\eta-1,\tau) + \\ \gamma g[x_0(\eta,\tau),\ x_0(\eta-1,\tau)] + Mx_0(\eta,\tau)\} + \\ F\cos\omega\,\eta \end{aligned} \qquad (2.37)$$

μ^2 terms :

$$\begin{aligned} x_2(\eta+1,\tau) - 2\cos\omega\ x_2(\eta,\tau) + x_2(\eta-1,\tau) = -\ 2x_1(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - \\ 2x_1(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) + \\ 2x_1(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + 2x_1(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2}) + Nx_1(\eta-1,\tau) + \\ Mx_1(\eta,\tau) + \text{coefficient of } \mu^2 \text{ in } g + 2\,\omega_1 \\ x_0(\eta+1,\tau) - (4\,\omega_1 + 2)\, x_0(\eta,\tau) + 2\,\omega_1 x_0(\eta-1,\tau) + \\ x_0(\eta,\tau+1) + x_0(\eta,\tau-1) + 2(s_1+N)[x_0(n+\tfrac{1}{2},\tau+\tfrac{1}{2}) - \\ x_0(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - x_0(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + x_0(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})]\} \end{aligned}$$

(2.38)

The solution to eqn. (2.36) in the generating or base solution and is obtained as follows :

The characteristic equation is

$$m^2 - 2\cos\omega\ m + 1 = 0$$

then the roots are

$$m_{1,2} = \frac{2\cos\omega \pm \sqrt{4\cos^2\omega - 4}}{2} = \frac{2\cos\omega \pm j\sqrt{4 - 4\cos^2\omega}}{2}$$

$$= \cos\omega \pm j\sin\omega$$

$$\text{Magnitude } R = \sqrt{\cos^2\omega + \sin^2\omega} = 1.0$$

$$\text{Phase } \theta = \tan^{-1}\left(\frac{\sin\omega}{\cos\omega}\right) = = \tan^{-1}(\tan\omega) = \omega$$

therefore the solution is

$$x_0(\eta,\tau) = A(\tau)\cos\omega\ \eta + B(\tau)\sin\omega\ \eta\ . \qquad (2.39)$$

$A(\tau)$ and $B(\tau)$ are constants for linear case, that is for $\mu = 0$ and are otherwise slowly varying amplitude functions depending on the slow time variable τ. It is of interest to obtain explicit expressions of the amplitude variations when the system is weakly nonlinear, that is $\mu \neq 0$. The method of obtaining the amplitude variations $A(\tau)$ and $B(\tau)$ by supressing secular terms generated on the right hand side of eqn. (2.37) proceeds as follows :

Let the difference interval be adjusted such that there exists some integer L such that

$$\omega = \frac{2\pi}{L} .$$

Expanding the nonlinear function $g(\cdot,\cdot)$ in eqn. (2.37) in a discrete fourier series [109] as :

$$g[x_o(\eta,\tau),\ x_o(\eta-1,\tau)] = \frac{C_o}{2} + \sum_{n=1}^{L/2} C_n \cos \frac{2\pi kn}{L} + \sum_{n=1}^{L/2} b_n \sin \frac{2\pi kn}{L} \qquad (2.40)$$

where

$$\frac{C_o}{2} = \frac{1}{L} \sum_{\eta=1}^{L} g(\cdot,\cdot)$$

$$c_n = \frac{2}{L} \sum_{\eta=1}^{L} g(\cdot,\cdot) \cos \frac{2\pi\eta n}{L}$$

$$b_n = \frac{2}{L} \sum_{\eta=1}^{L} g(\cdot,\cdot) \sin \frac{2\pi\eta n}{L} ,\ n = 1,2,3,\ldots L/2$$

for $n = 1$, c_1 and b_1 are the coefficients of $\cos \frac{2\pi\eta}{L}$, $\sin \frac{2\pi\eta}{L}$ respectively and are given by

$$c_1 = \frac{2}{L} \sum_{\eta=1}^{L} g(\cdot,\cdot) \cos \frac{2\pi\eta}{L}$$

$$b_1 = \frac{2}{L} \sum_{\eta=1}^{L} g(\cdot,\cdot) \sin \frac{2\pi\eta}{L} . \qquad (2.41)$$

Then eqn. (2.40) can be expressed with c_o, c_1 and b_1 as

$$g[x_o(\eta,\tau),\ x_o(\eta-1,\tau)] = \frac{c_o}{2} + c_1 \cos \frac{2\pi\eta}{L} + b_1 \sin \frac{2\pi\eta}{L}$$

$$= \frac{c_o}{2} + c_1 \cos \omega\eta + b_1 \sin\omega\eta . \qquad (2.42)$$

From eqn. (2.39)

$$x_0(\eta+\tfrac{1}{2},\ \tau+\tfrac{1}{2}) = [A(\tau+\tfrac{1}{2})\ \cos\tfrac{\omega}{2} + B(\tau+\tfrac{1}{2})\ \sin\tfrac{\omega}{2}]\ \cos\omega\ \eta$$

$$-\ [A(\tau+\tfrac{1}{2})\ \sin\tfrac{\omega}{2} - B(\tau+\tfrac{1}{2})\ \cos\omega/2]\ \sin\omega\,\eta$$

$$x_0(\eta+\tfrac{1}{2},\ \tau-\tfrac{1}{2}) = [A(\tau-\tfrac{1}{2})\ \cos\tfrac{\omega}{2} + B(\tau-\tfrac{1}{2})\ \sin\tfrac{\omega}{2}]\ \cos\omega\ \eta$$

$$-\ [A(\tau-\tfrac{1}{2})\ \sin\omega/2 - B(\tau-\tfrac{1}{2})\ \cos\omega/2]\ \sin\omega\,\eta$$

$$x_0(\eta-\tfrac{1}{2},\ \tau+\tfrac{1}{2}) = [A(\tau+\tfrac{1}{2})\ \cos\omega/2 - B(\tau+\tfrac{1}{2})\ \sin\omega/2]\ \cos\omega\,\eta$$

$$+\ [A(\tau+\tfrac{1}{2})\ \sin\ \omega/2 + B(\tau+\tfrac{1}{2})\ \cos\omega/2]\ \sin\omega\ \eta$$

$$x_0(\eta-\tfrac{1}{2},\ \tau-\tfrac{1}{2}) = [A(\tau-\tfrac{1}{2})\ \cos\omega/2 - B(\tau-\tfrac{1}{2})\ \sin\omega/2]\ \cos\omega\ \eta$$

$$+\ [A(\tau-\tfrac{1}{2})\ \sin\omega/2 + B(\tau-\tfrac{1}{2}\ \cos\omega/2]\ \sin\omega\ \eta$$

$$x_0(\eta-1,\tau) = [A(\tau)\ \cos\omega\ - B(\tau)\ \sin\omega\ ]\ \cos\omega\ \eta$$

$$+\ [A(\tau)\ \sin\omega\ + B(\tau)\ \cos\omega\ ]\ \sin\ \omega\ \eta\ . \qquad (2.43)$$

Substituting the eqns. (2.42) and (2.43) in (2.37) and supressing the secular terms by equating the coefficients of $\cos\omega\eta$ and $\sin\omega\eta$ to zero separately, the following equations for amplitude variations $A(\tau)$ and $B(\tau)$ are obtained :

$$\delta A(\tau) = \frac{1}{4\ \sin\omega/2}\ [MB(\tau) - b_1 + N(A(\tau)\ \sin\omega + B(\tau)\ \cos\omega\ )]$$

$$\delta B(\tau) = \frac{1}{4\ \sin\omega/2}\ [F + c_1 - MA(\tau) - N(A(\tau)\ \cos\omega - B(\tau)\ \sin\omega\ )]\ . \qquad (2.44)$$

Explicit expressions for $A(\tau)$ and $B(\tau)$ can be obtained from the above coupled equations if the system is linear or if the nonlinearity is of simple form. Such explicit expressions are however, not generally possible in most of the cases. Numerical techniques may then be resorted to get the required result. The solution to eqn. (2.44) and subsequent substitution in eqn. (2.39) provides the first order approximate solution and is given by

$$x(k) \simeq x_o(\eta,\tau) = A(\tau)\cos\omega\,\eta + B(\tau)\sin\omega\,\eta\ .$$

Then the eqn. (2.37) is solved with the remaining non-secular terms on the right hand side of the equation. Let the solution be

$$x_1(\eta,\tau) = C(\tau)\cos\omega\,\eta + D(\tau)\sin\omega\,\eta + \text{steady state solution due to nonsecular terms}$$

The amplitude variations $C(\tau)$ and $D(\tau)$ can be determined considering the eqn. (2.38) and supressing the secular terms arising in the right hand side due to substitution of $x_o(\eta,\tau)$ and $x_1(\eta,\tau)$. Knowing the complete solution of $x_1(\eta,\tau)$, now the approximate solution is given by

$$x(k) \simeq x_o(\eta,\tau) + \mu\, x_1(\eta,\tau)\ .$$

A similar procedure can be followed to include the higher order correction terms namely $x_2(\eta,\tau)$, $x_3(\eta,\tau)$ etc.

But the addition of the correction terms with the generating solution is possible when the system is linear or when the system is described with simple nonlinearities. The inclusion of first two correction terms, that is the approximate solution to the given equation upto the order of μ^2 is illustrated by considering a linear oscillator in example 2.7.1. The result obtained shows that the improved approximations are of course possible by considering higher order terms in the assumed series form.

The next section considers various examples, linear as well as nonlinear cases which will illustrate in detail the analysis technique as ewell as the applicability of the two variable expansion scheme to a clss of difference equations.

2.7 Numerical Examples :

The suitability of the proposed multiple discrete time perturbation technique to a general class of difference equations is illustrated in this section through some numerical examples. Linear and nonlinear equations are analysed and the results are compared with the exact solution of the given equation (simulated as a recurrence relation). The results obtained by the proposed method are sufficiently accurate for practical applications, however, a modified definition to the proposed scheme is given in the later part of this chapter. The modified definition is formulated out of practical

experience and this simplifies further the mathematical computations involved in the analysis of the proposed scheme. For convenience the following abbreviations are used for various schemes under study.

Scheme 1 - proposed scheme ;

Scheme 2 - Modified scheme (modified version of scheme 1)

Scheme 3 - The method proposed in [101] .

The results obtained by the Scheme 1, 2 and 3 are compared with the exact solution of the given system equation. The scheme 2 gives more accurate result for system whose solution is known to be bounded in addition to a reduction in involved mathematical calculations.

Example 2.7.1 : Linear (bounded/unbounded) systems :

Consider the following linear difference equation with the perturbation parameter μ

$$x(k+1) - x(k) + x(k-1) - \mu\, x(k-1) = 0 \qquad (2.45)$$

(i) for $0 < \mu < 1.0$, the solution decays to zero as $k \to \infty$

(ii) for $\mu < 0$ and $\mu > 1.0$, the response is unbounded, that is $x(k) \to \infty$ as $k \to \infty$

(iii) for $\mu = 1.0$ and for $\mu = 0.0$, the solution is oscillatory (bounded).

The application of perturbation methods for this type of small perturbation problems has been found to give fairly good approximation to the exact solution. The eqn. (2.45) will now be studied for small values of μ under unbounded as well as under damped conditions.

The direct application of the discrete two time scaling approach given in eqns. (2.28) and (2.29) to (2.45) gives

$$\begin{aligned}
&x(\eta+1,\tau) - x(\eta,\tau) + x(\eta-1,\tau) + \mu[2x(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) \\
&\quad - 2x(\eta+\tfrac{1}{2}, \tau-\tfrac{1}{2}) - 2x(\eta-\tfrac{1}{2}, \tau+\tfrac{1}{2}) + 2x(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2}) \\
&\quad - x(\eta-1,\tau)] + \mu^2\{ 2\omega_1\, x(\eta+1,\tau) - (4\,\omega_1+3)\, x(\eta,\tau) \\
&\quad + 2\omega_1\, x(\eta-1,\tau) + 2x(\eta,\tau+1) + x(\eta,\tau-1) + (2s_1-2) \\
&\qquad [x(\eta+\tfrac{1}{2}, \tau+\tfrac{1}{2}) - x(\eta+\tfrac{1}{2}, \tau-\tfrac{1}{2}) - x(\eta-\tfrac{1}{2}, \tau+\tfrac{1}{2}) \\
&\quad + x(\eta-\tfrac{1}{2}, \tau-\tfrac{1}{2})]\} + O(\mu^3) = 0 \; . \qquad (2.46)
\end{aligned}$$

Substituting eqn. (2.34) in eqn. (2.46) and collecting the coefficients of μ^0, μ^1, μ^2, ... and equating separately to zero, the following set of linear partial difference equations are obtained.

$$x_0(\eta+1, \tau) - x_0(\eta,\tau) + x_0(\eta-1, \tau) = 0 \qquad (2.47)$$

$$\begin{aligned}
&x_1(\eta+1,\tau) - x_1(\eta,\tau) + x_1(\eta-1,\tau) = -[2x_0(\eta+\tfrac{1}{2}, \tau+\tfrac{1}{2}) \\
&\quad - 2x_0(\eta+\tfrac{1}{2}, \tau-\tfrac{1}{2}) - 2x_0(\eta-\tfrac{1}{2}, \tau+\tfrac{1}{2}) + 2x_0(\eta-\tfrac{1}{2}, \tau-\tfrac{1}{2}) - x_0(\eta-1,\tau)]
\end{aligned} \qquad (2.48)$$

$$x_2(\eta+1,\ \tau) - x_2(\eta,\tau) + x_2(\eta-1,\ \tau) = -\ [2x_1(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2})$$

$$- 2x_1(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - 2x_1(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + 2x_1(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})$$

$$- x_1(\eta-1,\tau)] - \{\ 2\,\omega_1 x_0(\eta+1,\tau) - (4\,\omega_1+3)\ x_0(\eta,\tau)$$

$$+ 2\omega_1 x_0(\eta-1,\tau) + 2x_0(\eta,\tau+1) + x_0(\eta,\tau-1) + (2s_1-2)$$

$$[x_0(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - x_0(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - x_0(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2})$$

$$+ x_0(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})]\,\}\ . \qquad (2.49)$$

Solution to eqn. (2.47) is

$$x_0(\eta,\tau) = A(\tau)\ \cos\theta\eta + B(\tau)\ \sin\theta\eta \qquad (2.50)$$

where $A(\tau)$ and $B(\tau)$ are the slowly varying amplitude functions and θ is the natural frequency of the system. Here $\theta = \pi/3$.

Eqn. (2.50) is substituted on the right hand side of eqn. (2.48), using the relations given in eqn. (2.43), we get

$$x_1(\eta+1,\tau) - x_1(\eta,\tau) + x(\eta-1,\tau) = -2[A(\tau+\tfrac{1}{2})\ \cos\tfrac{\theta}{2}$$

$$+ B(\tau+\tfrac{1}{2})\ \sin\tfrac{\theta}{2}]\ \cos\theta\eta + 2[A(\tau+\tfrac{1}{2})\ \sin\tfrac{\theta}{2}$$

$$- B(\tau+\tfrac{1}{2})\ \cos\tfrac{\theta}{2}]\ \sin\theta\eta + 2[A(\tau-\tfrac{1}{2})\ \cos\tfrac{\theta}{2}$$

$$+ B(\tau-\tfrac{1}{2})\ \sin\tfrac{\theta}{2}]\ \cos\theta\eta - 2[A(\tau-\tfrac{1}{2})\ \sin\tfrac{\theta}{2}$$

$$- B(\tau-\tfrac{1}{2})\ \cos\tfrac{\theta}{2}]\ \sin\theta\eta + 2[A(\tau+\tfrac{1}{2})\ \cos\tfrac{\theta}{2}$$

$$- B(\tau+\tfrac{1}{2})\ \sin\tfrac{\theta}{2}]\ \cos\theta\eta + 2[A(\tau+\tfrac{1}{2})\ \sin\tfrac{\theta}{2}$$

$$+ B(\tau+\tfrac{1}{2})\ \cos\tfrac{\theta}{2}]\ \sin\theta\eta - 2[A(\tau-\tfrac{1}{2})\ \cos\tfrac{\theta}{2}$$

$$- B(\tau-\tfrac{1}{2})\ \sin\tfrac{\theta}{2}]\ \cos\theta\eta - 2[A(\tau-\tfrac{1}{2})\ \sin\tfrac{\theta}{2}$$

$$+ B(\tau-\tfrac{1}{2})\ \cos\tfrac{\theta}{2}]\ \sin\theta\eta + [A(\tau)\ \cos\theta - B(\tau)\ \sin\theta]$$

$$\cos\theta\eta + [A(\tau)\ \sin\theta + B(\tau)\ \cos\theta]\ \sin\theta\eta.$$

to eliminate the secular terms on the right hand side of the above eqn.

$$\text{coefficient of } \cos\theta\eta = 0$$

$$\text{and} \quad \text{coefficient of } \sin\theta\eta = 0$$

that is

$$4B(\tau+\tfrac{1}{2}) \sin \frac{\theta}{2} - 4B(\tau-\tfrac{1}{2}) \sin \frac{\theta}{2} = A(\tau) \cos \theta - B(\tau) \sin \theta$$

$$4A(\tau-\tfrac{1}{2}) \sin \frac{\theta}{2} - 4A(\tau-\tfrac{1}{2}) \sin \frac{\theta}{2} = -A(\tau) \sin \theta - B(\tau) \cos \theta$$

which can be rewritten as follows using the central difference operator δ,

$$\delta B(\tau) = \frac{+1}{4 \sin \frac{\theta}{2}} [A(\tau) \cos \theta - B(\tau) \sin \theta]$$

$$\delta A(\tau) = \frac{-1}{4 \sin \frac{\theta}{2}} [A(\tau) \sin \theta + B(\tau) \cos \theta] . \qquad (2.51)$$

The eqns. in (2.51) are first order linear coupled difference equations and these equations can be solved easily as follows. With $\theta = \pi/3$, the above equations can be rewritten as

$$\delta B(\tau) = \frac{1}{4} [A(\tau) - \sqrt{3}\, B(\tau)]$$

$$\delta A(\tau) = -\frac{1}{4} [\sqrt{3}\, A(\tau) + B(\tau)] . \qquad (2.52)$$

Differencing the first equation in (2.52) once

$$\delta^2 B(\tau) = \frac{1}{4} [\delta A(\tau) - \sqrt{3}\, \delta B(\tau)]$$

Substituting for $\delta B(\tau)$ from the second equation

$$\delta^2 B(\tau) = -\frac{\sqrt{3}}{4}\,\delta B(\tau) - \frac{1}{16}\,B(\tau) - \frac{\sqrt{3}}{16}\,A(\tau).$$

Again substituting for $A(\tau)$ from the first equation we obtain,

$$\delta^2 B(\tau) + \frac{\sqrt{3}}{2}\,\delta B(\tau) + \frac{1}{4}\,B(\tau) = 0\ . \tag{2.53}$$

As indicated in the remark at the end of the section 2.4, replacing

$$\delta^2 B(\tau) = B(\tau+1) - 2B(\tau) + B(\tau-1)$$

and $\delta B(\tau) = \dfrac{B(\tau+1) - B(\tau-1)}{2}$

the eqn. (2.53) can be rewritten as

$$B(\tau+1) - \left(\frac{7}{4+\sqrt{3}}\right) B(\tau) + \left(\frac{4-\sqrt{3}}{4+\sqrt{3}}\right) B(\tau-1) = 0\ . \tag{2.54}$$

The solution of (2.54) is of the form

$$B(\tau) = R^{\tau}[P\cos\varphi\tau + Q\sin\varphi\tau] \tag{2.55}$$

where

$$R^2 = \frac{4-\sqrt{3}}{4+\sqrt{3}}$$

$$\varphi = \tan^{-1}\left[\frac{\sqrt{\left(\frac{4-\sqrt{3}}{4+\sqrt{3}}\right) - \left(\frac{3.5}{4+\sqrt{3}}\right)^2}}{3.5/(4+\sqrt{3})}\right]$$

and P,Q are the constants to be determined from the initial conditions.

Then from the first expression in (2.52)

$$A(\tau) = 4\delta B(\tau) + \sqrt{3}\ B(\tau).$$

Substituting for $B(\tau)$ and $\delta B(\tau)$ from the eqn. (2.55), the following expression for $A(\tau)$ is obtained.

$$A(\tau) = R^{\tau}\ [Q \cos \varphi\tau - P \sin \varphi\tau]. \tag{2.56}$$

Combining the equations (2.56), (2.55) and (2.50) we obtain

$$x_0(\eta,\tau) = R^{\tau}[A \cos(\theta\eta - \varphi\tau) + B \sin(\theta\eta - \varphi\tau)]$$

Hence the approximate solution to the given equation upto the order μ is

$$x(k) \approx x_0(\eta,\tau) = R_1^k\ [A \cos \theta_1 k + B \sin \theta_1 k] \tag{2.57}$$

where $R_1 = R^{\mu}$

$$\theta_1 = (\theta - \mu\varphi).$$

Figures 2.2(a) and 2.2(b) give the comparison of the approximate solution given in (2.57) with the exact solution of the given equation for $\mu = 0.1$ and $\mu = 0.3$. It is observed that the approximate solution is closer to the exact solution for small μ, namely for $\mu = 0.1$ for all time and the deviation of the results increases as k increases for larger μ. This situation is shown in Fig. 2.2(b).

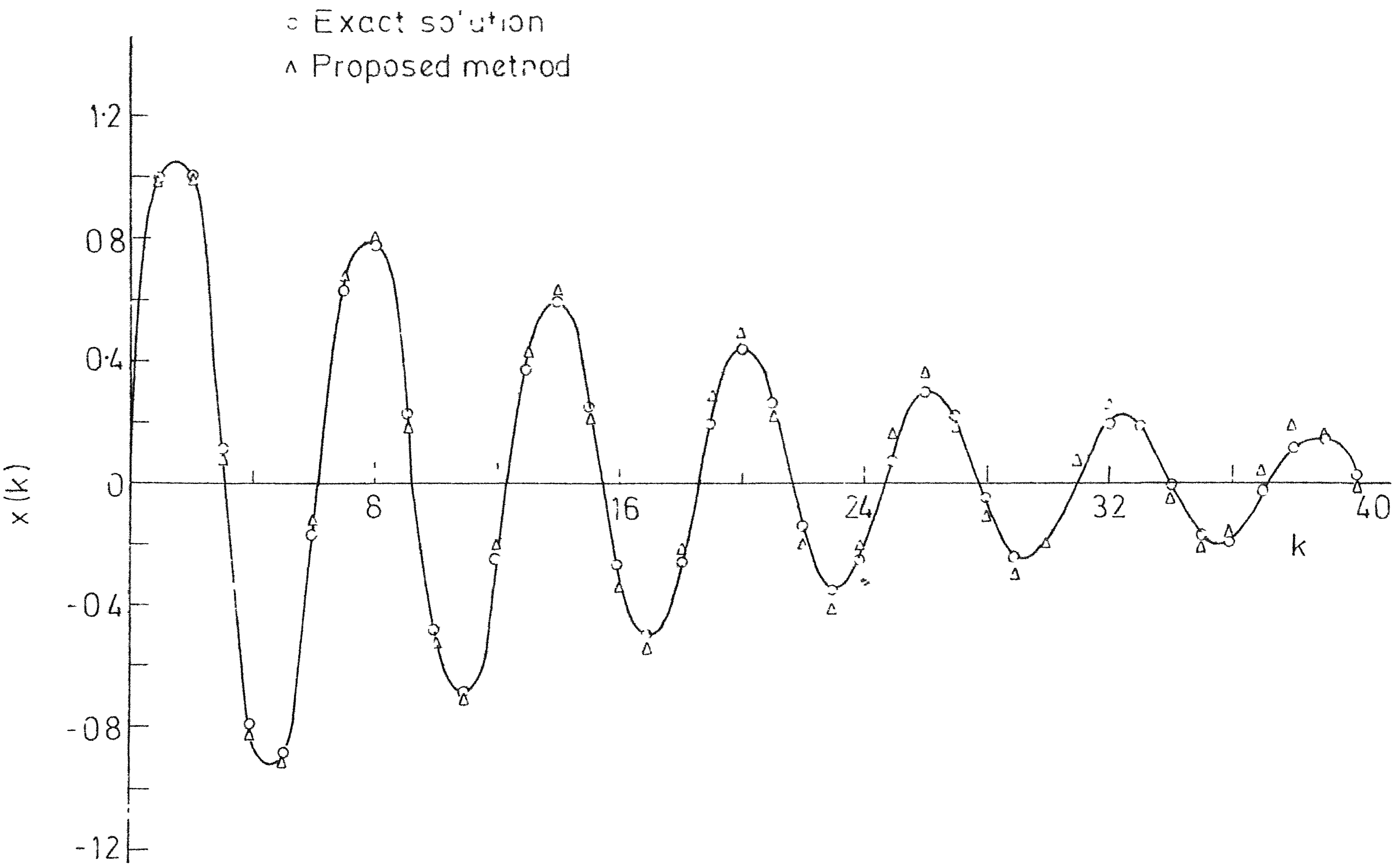


FIG 22a LINEAR UNDER DAMPED SYSTEM (μ=0·1)

$x(k+1)-x(k)+x(k-1)-\mu x(k-1)=0$

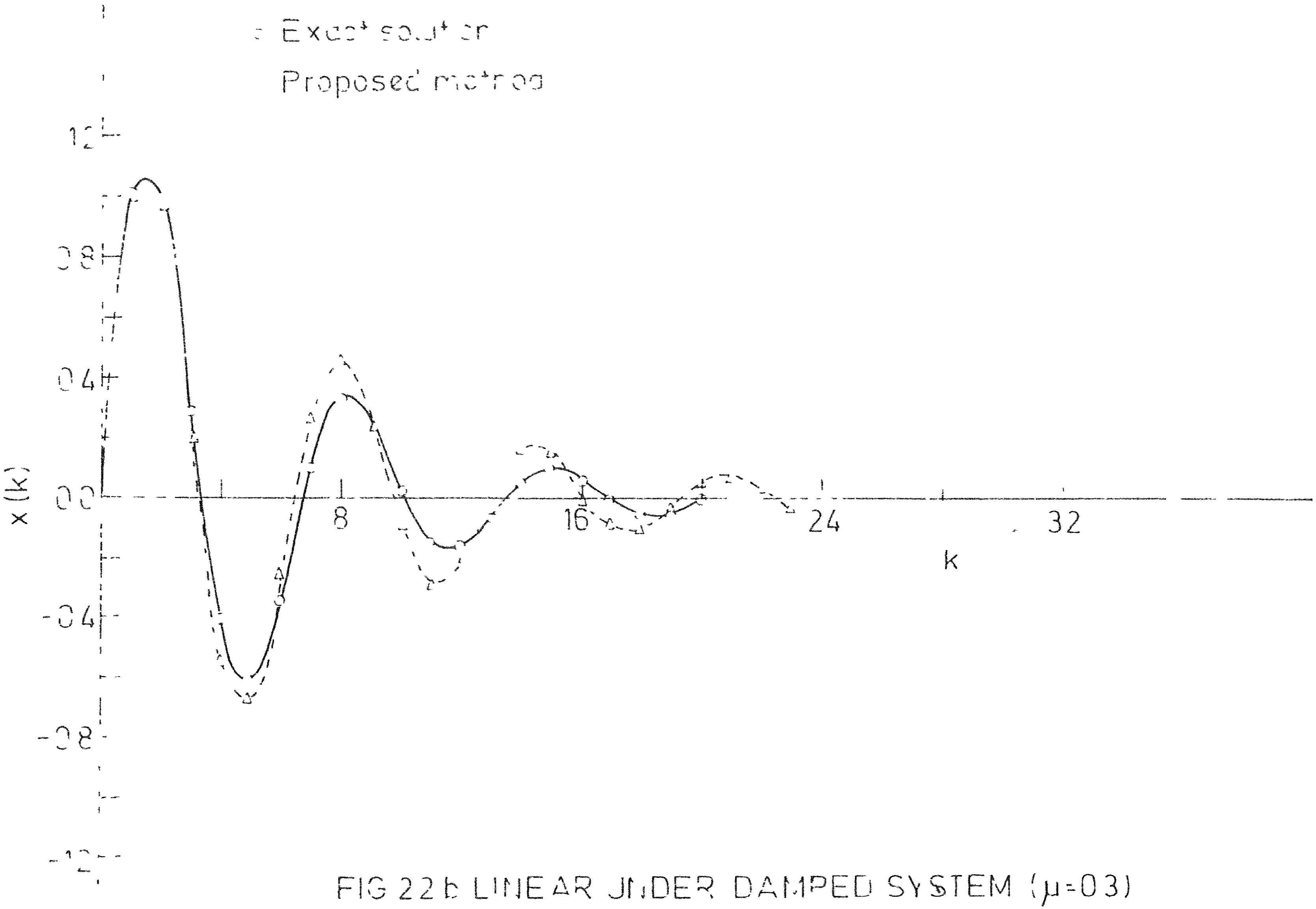


FIG 2.2 b LINEAR UNDER DAMPED SYSTEM (μ=0.3)

$x(k+1)-x(k)+x(k-1)-\mu x(k-1)=0$

Figures 2.3(a) and 2.3(b) show the unbounded response of a linear difference equation. Here again for small μ the approximate result obtained by the proposed method is closer to the exact solution For large μ an appreciable diviation of the results is evident

Now, the procedure to obtain the second correction term will be considered. Before proceeding with the calculations for second correction term it will be useful to say something about the initial conditions due to introduction of two independent time factors.

Let $x(o) = x_o$ and $x(1) = x_1$ be given initial conditions for the second order equation given in (2.45). Then from eqns. (2.27) and (2.34) we obtain the following initial conditions.

(i) $x_o(0,0) = x_o$

$x_o(1,0) = x_1$

(ii) $x_1(0,0) = 0$

$x_1(1,0) + x_o(0,1) - x_o(0,0) = 0$

(iii) $x_2(0,0) = 0$

$$x_2(1,0) + x_1(0,1) - x_1(0,0) + \omega_1 x_o(1,0) + s_1 x_o(0,1) - (\omega_1 + s_1)\, x_o(0,0) = 0$$

and so on.

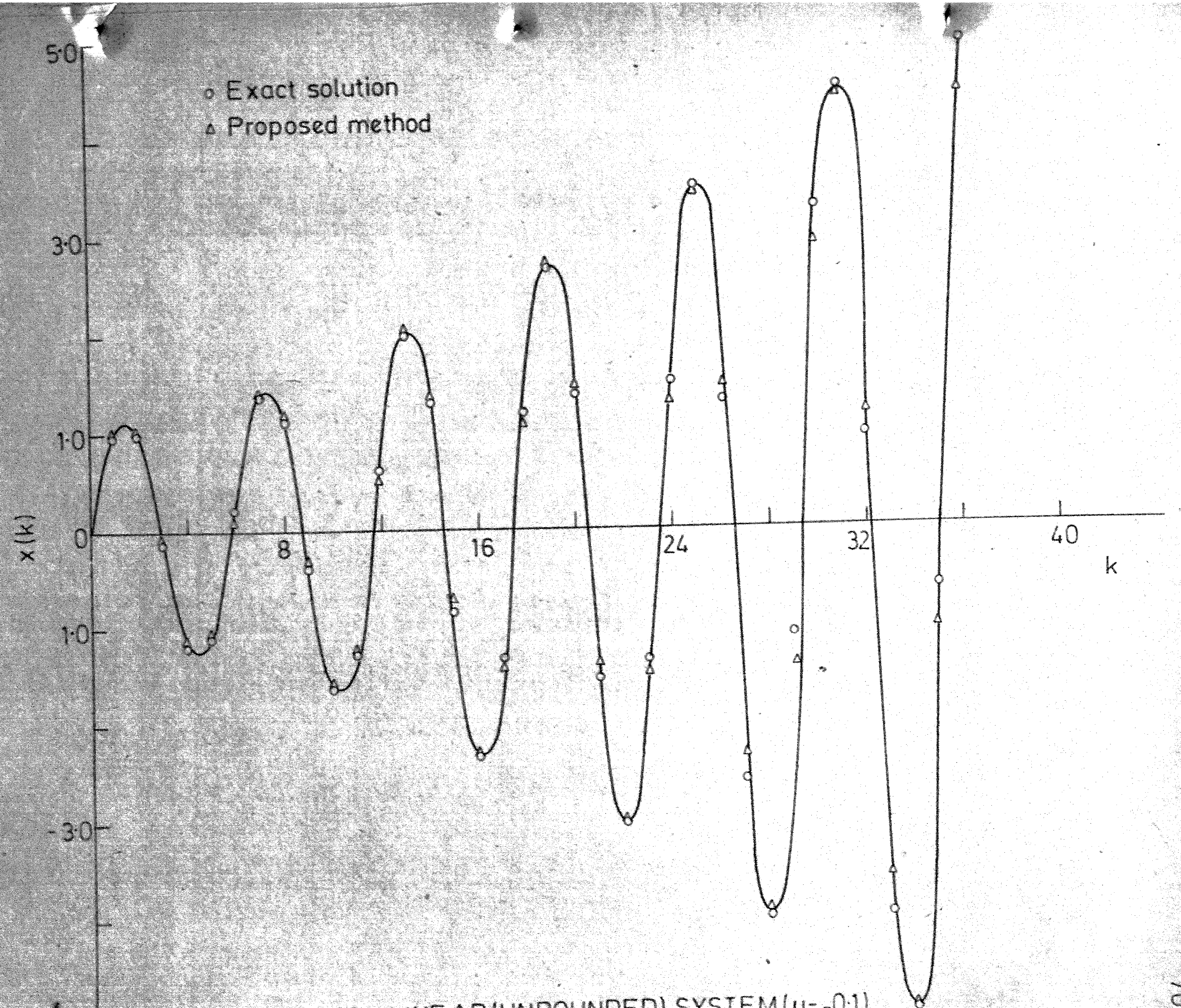


FIG.23a LINEAR (UNBOUNDED) SYSTEM (μ=-0·1)
$x(k+1)-x(k)+x(k-1)-\mu x(k-1)=0$

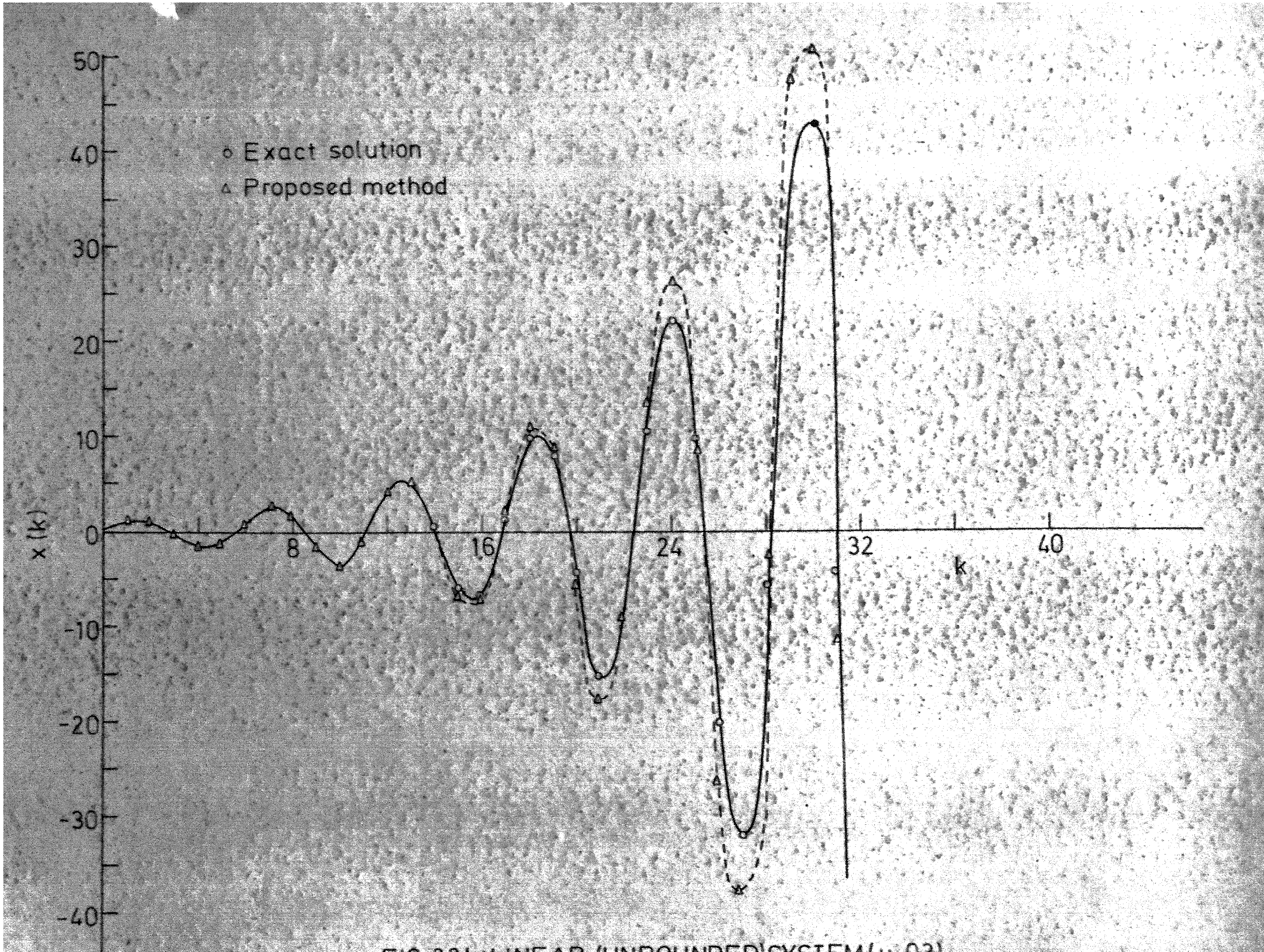


FIG 2.3b LINEAR (UNBOUNDED) SYSTEM (μ=-0.3)

$x(k+1)-x(k)+x(k-1)-\mu x(k-1)=0$

With $x_o = 0.0$ and $x_1 = 1.0$, the generating solution for $\mu = 0.1$, becomes

$$x_o(\eta,\tau) = \frac{2}{\sqrt{3}} (0.63)^{\tau} \sin(1.0?7\ k) \qquad (2.58)$$

where

$$\tau = 0.1\ k.$$

Then the solution to eqn. (2.48) is given by

$$x_1(\eta,\tau) = C(\tau) \cos\theta\eta + D(\tau) \sin\theta\eta\ . \qquad (2.59)$$

Substitution of eqns. (2.58) and (2.59) on the right hand side of eqn. (2.49) leads to

$$\begin{aligned} x_2(\eta+1,\tau) - x_2(\eta,\tau) + x_2(\eta-1,\tau) &= -2[C(\tau+\tfrac{1}{2}) \cos \tfrac{\theta}{2}\ D(\tau+\tfrac{1}{2})\sin \tfrac{\theta}{2}] \\ &\quad \cos\theta\eta + 2\ [C\ (\tau+\tfrac{1}{2}) \sin \tfrac{\theta}{2} - D(\tau+\tfrac{1}{2}) \cos \tfrac{\theta}{2}] \sin\theta\eta \\ &+ 2[C(\tau-\tfrac{1}{2}) \cos \tfrac{\theta}{2} + D(\tau-\tfrac{1}{2}) \sin \tfrac{\theta}{2}] \cos\theta\eta - 2[C(\tau-\tfrac{1}{2}) \sin \tfrac{\theta}{2} \\ &- D(\tau-\tfrac{1}{2}) \cos \tfrac{\theta}{2}] \sin\theta\eta + 2[C(\tau+\tfrac{1}{2}) \cos \tfrac{\theta}{2} - D(\tau+\tfrac{1}{2}) \sin \tfrac{\theta}{2}] \\ &\quad \cos\theta\eta + 2[C(\tau+\tfrac{1}{2}) \sin \tfrac{\theta}{2} + D(\tau+\tfrac{1}{2}) \cos \tfrac{\theta}{2}] \sin\theta\eta \\ &- 2[C(\tau-\tfrac{1}{2}) \cos \tfrac{\theta}{2} - D(\tau-\tfrac{1}{2}) \sin \tfrac{\theta}{2}] \cos\theta\eta - 2[C(\tau-\tfrac{1}{2}) \sin \tfrac{\theta}{2} \\ &+ D(\tau-\tfrac{1}{2}) \cos \tfrac{\theta}{2}] \sin\theta\eta + [C(\tau) \cos \theta - D(\tau) \sin\theta] \cos\theta\eta \\ &+ [A(\tau) \sin \theta + B(\tau) \cos \theta] \sin\theta\eta - 2\ \omega_1[R^{\tau} \cos\varphi\tau \\ &- R^{\tau}/\sqrt{3} \sin\varphi\tau] \cos\theta\eta - 2\ \omega_1[\tfrac{R^{\tau}}{\sqrt{3}} \cos\varphi\tau + R^{\tau} \sin\varphi\tau] \sin\theta\eta \end{aligned}$$

$$- (4\omega_1 + 3)\frac{2}{\sqrt{3}} R^{\tau} \sin\varphi\tau \cos\theta\eta + (4\omega_1+3)$$

$$\frac{2}{\sqrt{3}} R^{\tau} \cos\varphi\tau \sin\theta\eta + 2\omega_1\left[\frac{R^{\tau}}{\sqrt{3}} \sin\varphi\tau + R^{\tau} \cos\varphi\tau\right]$$

$$\cos\theta\eta - 2\omega_1\left[\frac{R^{\tau}}{\sqrt{3}} \cos\varphi\tau - R^{\tau} \sin\varphi\tau\right] \sin\theta\eta +$$

$$\frac{4R}{\sqrt{3}} R^{\tau} \sin(\varphi\tau + \varphi) \cos\theta\eta - \frac{4R}{\sqrt{3}} R^{\tau} \cos(\varphi\tau+\varphi) \sin\theta\eta +$$

$$\frac{2}{\sqrt{3R}} R^{\tau} \sin(\varphi\tau - \varphi) \cos\theta\eta - \frac{2}{\sqrt{3R}} R^{\tau} \cos(\varphi\tau-\varphi) \sin\theta\eta +$$

$$(2-2s_1)\left[\frac{2\sqrt{R}}{\sqrt{3}} R^{\tau} \sin\left(\frac{\theta}{2} - \varphi\tau - \frac{\varphi}{2}\right) - \frac{2}{\sqrt{3R}} R^{\tau} \sin\left(\frac{\theta}{2}-\varphi\tau+\frac{\varphi}{2}\right) +\right.$$
$$\left.\frac{2\sqrt{R}}{\sqrt{3}} R^{\tau} \sin\left(\frac{\theta}{2}+\varphi\tau+\frac{\varphi}{2}\right) - \frac{2}{\sqrt{3R}}R^{\tau} \sin\left(\frac{\theta}{2}+\varphi\tau-\frac{\varphi}{2}\right)\right]\sin\theta\eta+$$
$$\left[\frac{2\sqrt{R}}{\sqrt{3}}R^{\tau} \cos\left(\frac{\theta}{2} - \varphi\tau - \frac{\varphi}{2}\right) - \frac{2}{\sqrt{3R}} R^{\tau} \cos\left(\frac{\theta}{2} - \varphi\tau + \frac{\varphi}{2}\right) -\right.$$
$$\left.\frac{2\sqrt{R}}{\sqrt{3}} R^{\tau} \cos\left(\frac{\theta}{2} + \varphi\tau + \frac{\varphi}{2}\right) + \frac{2}{\sqrt{3R}} R^{\tau} \cos\left(\frac{\theta}{2} + \varphi\tau - \frac{\varphi}{2}\right)\right] \sin\theta\eta.$$

Elimination of secular terms results in the following equations.

$$2\omega_1 + \frac{2R}{\sqrt{3}} \sin\varphi - \frac{1}{\sqrt{3R}} \sin\varphi + (1 - s_1) \left[\frac{2R}{\sqrt{3}} \sin\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)\right.$$

$$\left. - \frac{2}{\sqrt{3R}} \sin\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) + \frac{2\sqrt{R}}{\sqrt{3}} \sin\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) - \frac{2}{\sqrt{3R}} \sin\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)\right] = 0$$

$$\frac{2\omega_1}{\sqrt{3}} - \left(\frac{4\omega_1 + 3}{\sqrt{3}}\right) + \frac{2R}{\sqrt{3}} \cos\varphi + \frac{1}{\sqrt{3R}} \cos\varphi + (1 - s_1)$$

$$\left[- \frac{2\sqrt{R}}{\sqrt{3}} \cos\left(\frac{\theta}{2} - \frac{\varphi}{2}\right) + \frac{2}{\sqrt{3R}} \cos\left(\frac{\theta}{2} + \frac{\varphi}{2}\right) + \frac{2\sqrt{R}}{\sqrt{3}} \cos\left(\frac{\theta}{2} + \frac{\varphi}{2}\right)\right.$$

$$\left.- \frac{2}{\sqrt{3R}} \cos\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)\right] = 0 .$$

These algebraic equations involving the two unknowns can now be solved for ω_1, s_1 and

$$\omega_1 = -0.05$$

$$s_1 = 1.247 .$$

It is to be noted that the contribution due to second correction term is more significant in the slow time factor τ than the fast time factor η since $s_1 >> \omega_1$. This contradicts the statement given by Morrison [104], for continuous time system namely the inclusion of the constants S_n in the definition of slow time scale being to account for the general dependence of the nonlinear function upon μ.

Table 2.1 shows the results of the linear discrete system described by eqn. (2.45) upto the order μ^2. It is observed that the solution upto second correction terms is closer to the exact solution. This tabulated results are obtained for $\mu = 0.1$ in the eqn. (2.45).

Table 2.1

Solution of equation (2.45) with $\mu = 0.1$

Discrete time instant k	Exact solution	Solution upto the order μ	Solution upto the order μ^2
1	1.000	1.000	1.000
2	1.000	0.995	0.995
3	0.100	0.078	0.088
4	-0.800	-0.829	-0.808

contd....

Discrete time instant k	Exact solution	Solution upto the order μ	Solution upto the order μ^2
5	-0.890	-0.896	-0.883
6	-0.170	0.135	-0.150
7	0.631	0.682	0.645
8	0.784	0.801	0.778
9	0.216	0.175	0.192
10	-0.490	0.556	-0.511
11	-0.684	-0.713	-0.681
12	-0.243	-0.202	-0.217
13	0.372	0.449	0.398
14	0.591	0.630	0.591
15	0.256	0.218	0.229
16	-0.276	-0.358	-0.304
17	-0.506	-0.555	-0.509
18	-0.258	-0.225	-0.232
19	0.198	0.281	0.228
20	0.430	0.485	0.436
21	0.252	0.226	0.228
22	-0.135	-0.217	-0.166
23	-0.362	-0.422	-0.370
24	-0.240	0.222	-0.219
25	0.085	0.164	0.116
26	0.302	0.365	0.312
27	0.225	0.214	0.207
28	-0.046	0.120	-0.076
29	-0.249	-0.315	-0.262
30	-0.207	-0.203	-0.192
31	0.017	0.085	0.045
32	0.203	0.269	0.218
33	0.188	0.191	0.176
34	0.005	-0.056	-0.021
35	-0.169	-0.229	-0.179

In the following section a linear oscillatory type of system with small perturbation is considered and the analysis is carried out with the proposed scheme and the results are verified by computer simulation.

Example 2.7.2 : Linear discrete oscillator :

Consider the linear discrete oscillator described by :

$$x(k+1) - x(k) + x(k-1) + \beta\, x(k) = 0 \tag{2.60}$$

Depending on the numerical value of β, this equation can be rewritten as shown below.

(i) for valves $-1.0 < \beta < 1.0$, β can be taken as a small parameter μ, that is $\mu = \beta$ and the equation (2.60) assumes the form

$$x(k+1) - x(k) + x(k-1) + \mu\, x(k) = 0 \tag{2.61}$$

(ii) for values $1 < \beta < 3$, β can be rewritten as $\beta = \alpha + \mu$, so that $\mu = \beta - \alpha$ is always less than unity. Under this condition the system equation can be rewritten as

$$x(k+1) - (1-\alpha)\, x(k) + x(k-1) + \mu\, x(k) = 0\,. \tag{2.62}$$

Now the eqns. (2.61) and (2.62) are of the form to apply two variable expansion scheme. Considering the eqn. (2.61), the basic solution is

$$x_o(\eta,\tau) = A(\tau)\cos\theta\eta + B(\tau)\sin\theta\eta \tag{2.63}$$

with $\theta = \pi/3$.

The right hand side of difference equation relating x_1 (first correction term) is

$$= 2x_o(\eta+\tfrac{1}{2},\ \tau+\tfrac{1}{2}) - 2x_o(\eta+\tfrac{1}{2},\ \tau-\tfrac{1}{2}) - 2x_o(\eta-\tfrac{1}{2},\ \tau+\tfrac{1}{2})$$

$$+ 2x_o(\eta-\tfrac{1}{2},\ \tau-\tfrac{1}{2}) + x_o(\eta,\tau). \qquad (2.64)$$

Substituting (2.63) in (2.64) and equating the coefficients of $\cos\theta\eta$ and $\sin\theta\eta$ separately to zero, the following equations are obtained.

$$\delta B(\tau) = \frac{-1}{4\sin\frac{\theta}{2}}\ A(\tau) \qquad (2.65)$$

$$\delta A(\tau) = \frac{1}{4\sin\frac{\theta}{2}}\ B(\tau)\ . \qquad (2.66)$$

Differencing eqn. (2.65) once on both sides with δ

$$\delta^2 B(\tau) = \frac{-1}{4\sin\frac{\theta}{2}}\ \delta A(\tau)\ .$$

Combining the eqn. (2.66) with the above equation, we get

$$\delta^2 B(\tau) + \frac{B(\tau)}{16\sin^2\frac{\theta}{2}} = 0\,.$$

With $\theta = \pi/3$, and expanding, the following linear difference equation is obtained

$$B(\tau+1) - 1.75\ B(\tau) + B(\tau-1) = 0. \qquad (2.67)$$

The solution to the above equation is of the form

$$B(\tau) = p \cos\varphi\tau + q \sin\varphi\tau \tag{2.68}$$

where

$$\varphi = \tan^{-1}\left[\frac{\sqrt{1-(\frac{1.75}{2})^2}}{(1.75/2)}\right],$$

and p and q are the constants to be evaluated knowing the initial conditions. Then from eqn. (2.65)

$$A(\tau) = p \sin\varphi\tau - q \cos\varphi\tau . \tag{2.69}$$

From equations (2.68), (2.69) and (2.63), the approximate solution upto the order μ is given by

$$x_0(\eta,\tau) = A \cos\theta_1 k + B \sin\theta_1 k \tag{2.70}$$

where $\theta_1 = \theta + \varphi\mu$ and A and B are the constants.

The solution given in eqn. (2.70) is compared with the exact solution of the given equation in figures 2.4(a) to 2.4(d) for various values of μ.

It is observed that the phase difference between the solutions increases as μ increases, but the magnitude of the solution remains unaltered.

Example 2.7.3 : Weakly nonlinear difference equation

In this example a discrete time system described by a weakly nonlinear difference equation is considered. The discrete version of the Duffing equation with weak forcing

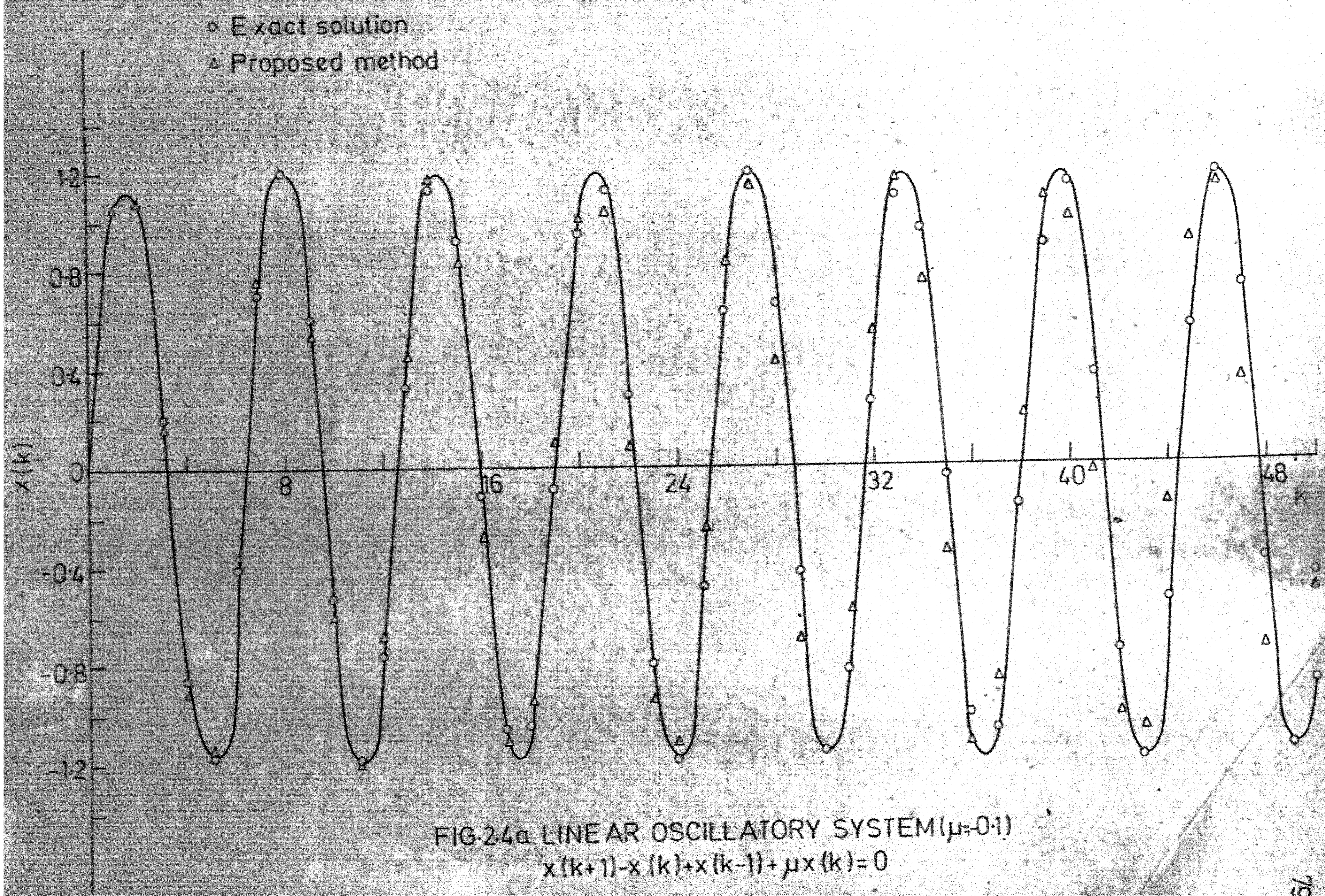


FIG.2.4a LINEAR OSCILLATORY SYSTEM (μ=-0·1)

$x(k+1)-x(k)+x(k-1)+\mu x(k)=0$

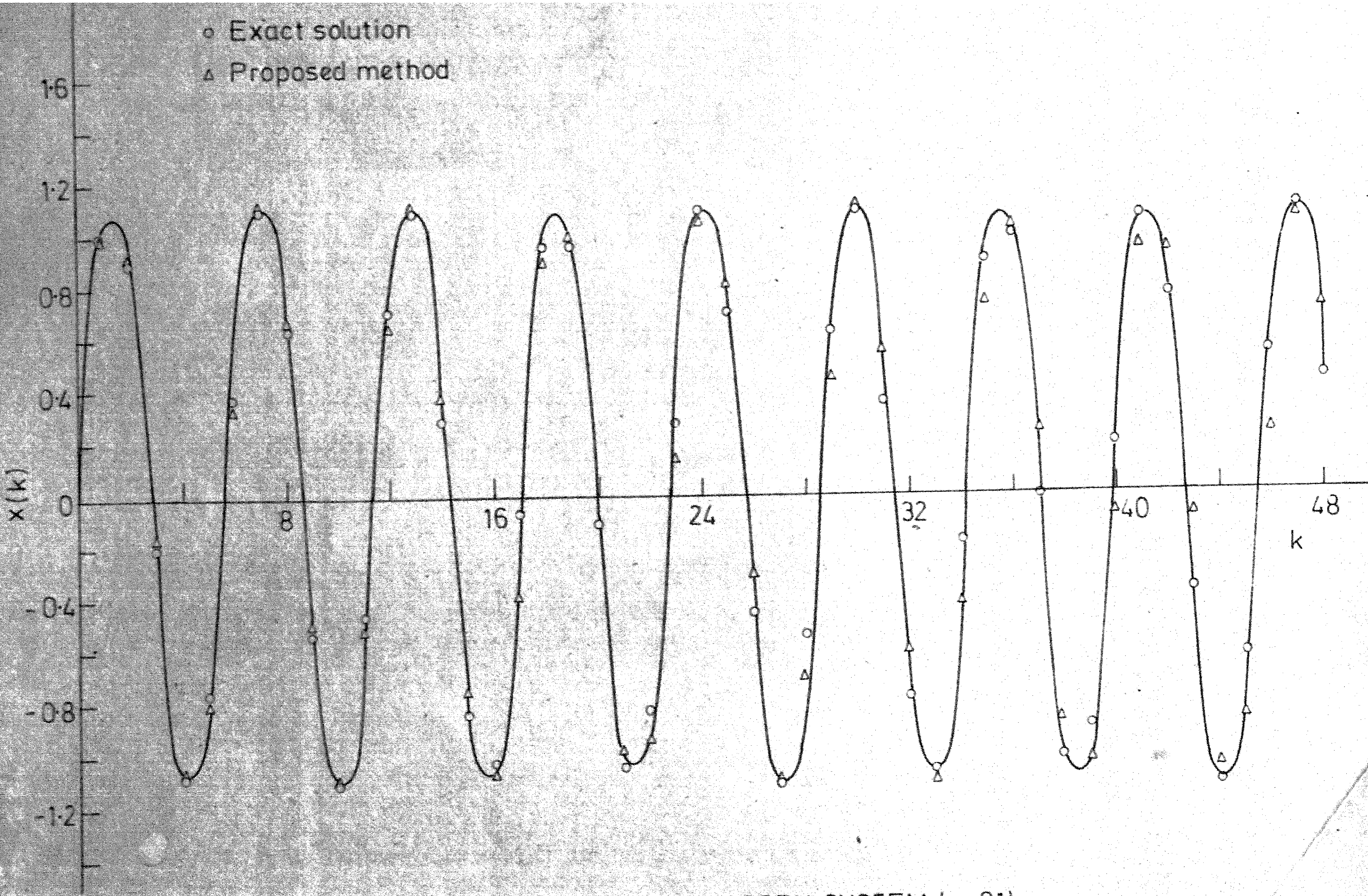


FIG.2·4 b LINEAR OSCILLATORY SYSTEM (μ=0·1)

$x(k+1)-x(k)+x(k-1)+\mu x(k)=0$

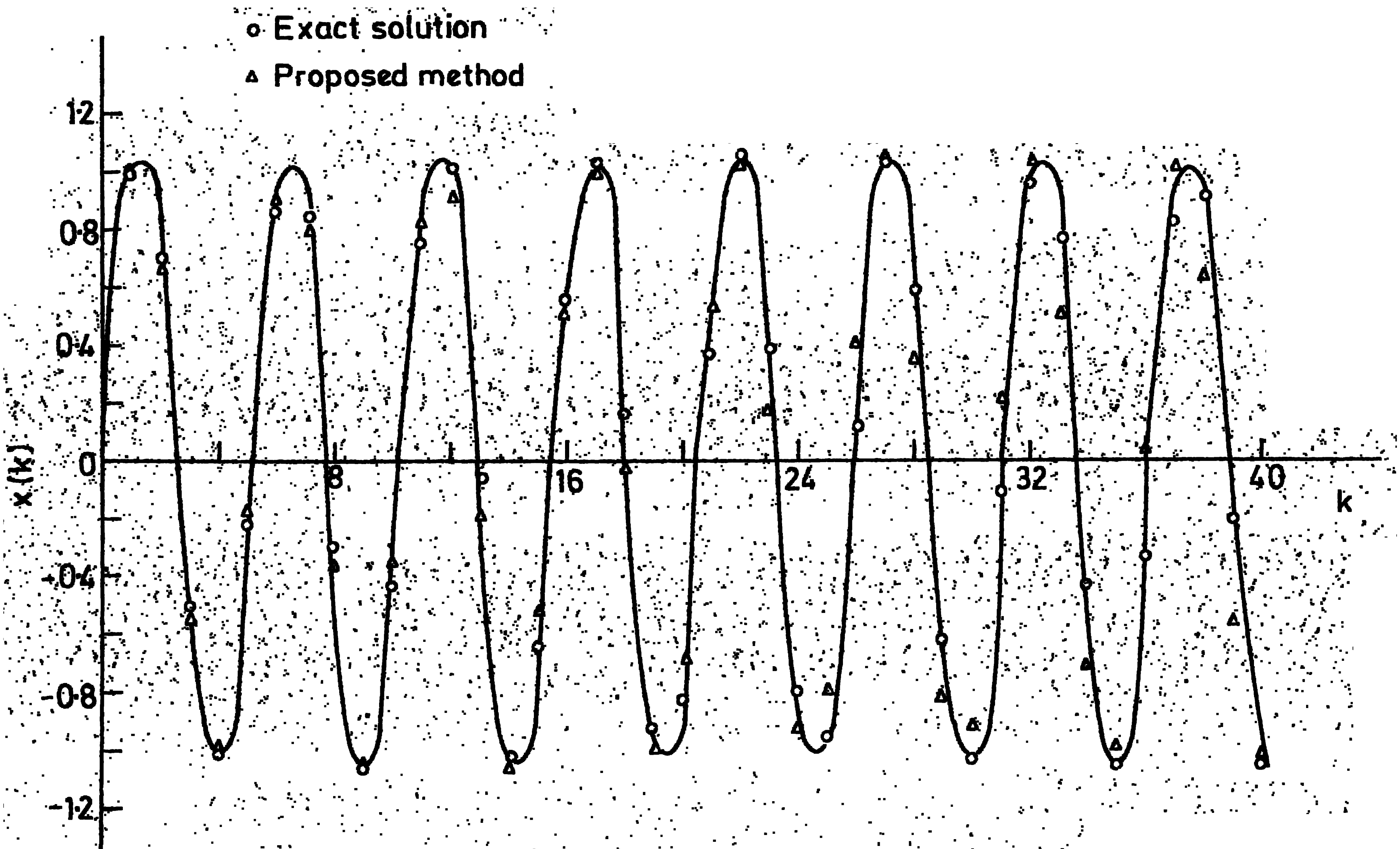


FIG-2.4.c LINEAR OSCILLATORY SYSTEM ($\mu = 0.3$)

$x(k+1) - x(k) + x(k-1) + \mu x(k) = 0$

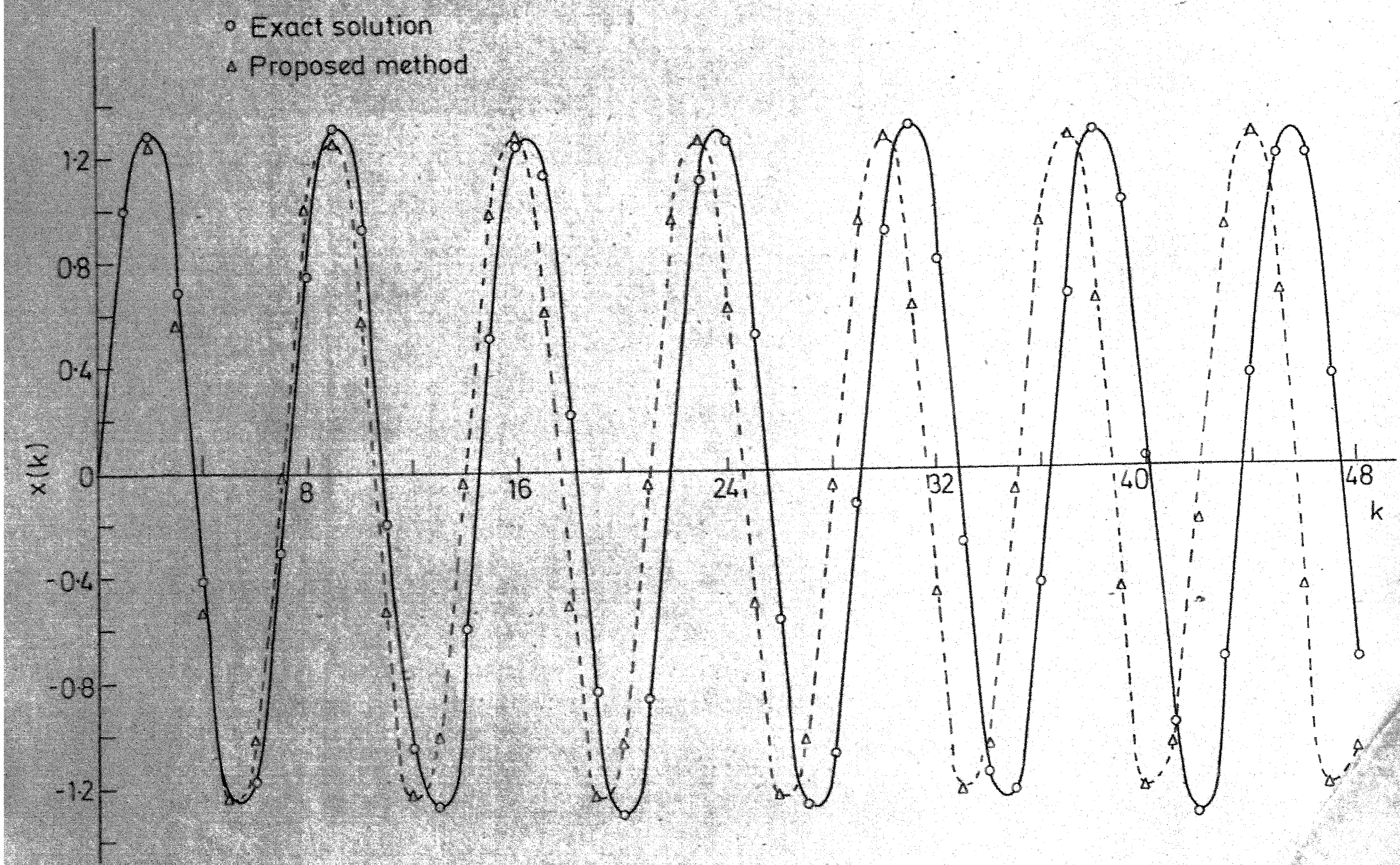


FIG-2·4d LINEAR OSCILLATORY SYSTEM (μ=-0·3)

$x(k+1)-x(k)+x(k-1)+\mu x(k)=0$

is under investigation. Note that this enables one to make a comparison of the discrete model analysis with the continuous time system solutions. The example (a) discusses the situation where the continuous time version is dissipation free and the example (b) treats the case where there is weak viscous damping. The discrete models are derived in Appendix B and the analysis of these models are carried out here.

(a) Duffing Equation (Dissipation free):

The discrete model is

$$x(k+1) + \lambda x(k) + x(k-1) + \mu \gamma x^3(k) = \mu F \cos \omega k \, . \quad (2.71)$$

Adding and subtracting $2\cos \omega \; x(k)$ in the above equation and rewriting.

$$x(k+1) - 2 \cos\omega \; x(k) + x(k-1) = \mu[F \cos\omega k - \gamma x^3(k) - Mx(k)] \quad (2.72)$$

where M is the detuning parameter and is defined as

$$\mu M \triangleq \lambda + 2 \cos \omega \, .$$

It is to be noted that the addition of the term $2\cos \omega \; x(k)$ in the above equation assures bounded linear response.

The application of the two time perturbation procedure to eqn. (2.72) leads to the following set of linear difference equations

$$x_0(\eta+1,\tau) - 2 \cos \omega \; x_0(\eta,\tau) + x_0(\eta-1,\tau) = 0 \quad (2.73)$$

$$x_1(\eta+1,\tau) - 2\cos\omega\ x_1(\eta,\tau) + x_1(\eta-1,\tau) = F\cos\omega\eta - \gamma x_0^3(\eta,\tau)$$
$$- Mx_0(\eta,\tau) - 2[x_0(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - x_0(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2})$$
$$- x_0(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + x_0(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})] \, . \qquad (2.74)$$

The solution $x_0(\eta,\tau)$ obtained from equation (2.73) is

$$x_0(\eta,\tau) = A(\tau)\cos\omega\eta + B(\tau)\sin\omega\eta$$

Substitution of the generating solution using (2.43) in the right hand ~~hand~~ side of eqn. (2.74) yields

$$x_1(\eta+1,\tau) - 2\cos\omega\ x_1(\eta,\tau) + x_1(\eta-1,\tau) = F\cos\omega\eta$$
$$- \gamma[A(\tau)\cos\omega\eta + B(\tau)\sin\omega\eta]^3 - M[A(\tau)\cos\omega\eta$$
$$+ B(\tau)\sin\omega\eta] - 2[A(\tau+\tfrac{1}{2})\cos\tfrac{\omega}{2} + B(\tau+\tfrac{1}{2})\sin\tfrac{\omega}{2}]\cos\omega\eta$$
$$- 2[A(\tau+\tfrac{1}{2})\sin\tfrac{\omega}{2} - B(\tau+\tfrac{1}{2})\cos\tfrac{\omega}{2}]\sin\omega\eta + 2[A(\tau-\tfrac{1}{2})\,.$$
$$\cos\omega/2 + B(\tau-\tfrac{1}{2})\sin\omega/2]\cos\omega\eta - 2[A(\tau-\tfrac{1}{2})\sin\omega/2$$
$$- B(\tau-\tfrac{1}{2})\cos\omega/2]\sin\omega\eta + 2[A(\tau+\tfrac{1}{2})\cos\omega/2$$
$$- B(\tau+\tfrac{1}{2})\sin\omega/2]\cos\omega\eta + 2[A(\tau+\tfrac{1}{2})\sin\omega/2$$
$$+ B(\tau+\tfrac{1}{2})\cos\omega/2]\sin\omega\eta - 2[A(\tau-\tfrac{1}{2})\cos\omega/2$$
$$- B(\tau-\tfrac{1}{2})\sin\omega/2]\cos\omega\eta - 2[A(\tau-\tfrac{1}{2})\sin\omega/2$$
$$+ B(\tau-\tfrac{1}{2})\cos\omega/2]\sin\omega\eta \qquad (2.75)$$

In the above, nonlinear term can be rewritten as

$$[A(\tau)\cos\omega\eta + B(\tau)\sin\omega\eta]^3 = [\tfrac{3}{4}A^3(\tau) + \tfrac{3}{4}A(\tau)B^2(\tau)]\cos\omega\eta$$
$$+ [\tfrac{3}{4}B^3(\tau) + \tfrac{3}{4}A^2(\tau)\,B(\tau)]\sin\omega\eta$$
$$+ \text{Non-secular terms}. \qquad (2.76)$$

Substituting the eqn. (2.76) in eqn. (2.75) and collecting the coefficients of $\cos\omega\eta$ and $\sin\omega\eta$ and equating them separately to zero the following equations for $A(\tau)$ and $B(\tau)$ are obtained ;

$$\delta A(\tau) = \frac{1}{4\sin\omega/2}[\tfrac{3}{4}\gamma B(\tau)\ (A^2(\tau) + B^2(\tau)) + M\,B(\tau)]$$

$$\delta B(\tau) = \frac{1}{4\sin\omega/2}[F - \tfrac{3}{4}\gamma A(\tau)\ (A^2(\tau) + B^2(\tau)) - M\,A(\tau)] \qquad (2.77)$$

where $\delta A(\tau) = A(\tau + \tfrac{1}{2}) - A(\tau - \tfrac{1}{2})$ and likewise $\delta B(\tau)$.

If a steady state solution for $A(\tau)$ and $B(\tau)$ exists then

$$\delta A(\tau) = \delta B(\tau) = 0$$

and $A(\tau) = A$, $B(\tau) = B$

where A and B are the steady state values of $A(\tau)$ and $B(\tau)$ respectively.

Then under steady state condition

$$B[\tfrac{3}{4}\gamma\ (A^2 + B^2) + M] = 0 \qquad (2.78)$$

$$F - A\,[\tfrac{3}{4}\gamma\ (A^2 + B^2) + M] = 0\ . \qquad (2.79)$$

From eqn. (2.78)

$$\text{either } B = 0$$

$$\text{or } \quad M = -\frac{3}{4}\gamma\,(A^2 + B^2)\,.$$

Combining the second condition with eqn. (2.79), we get

$F = 0$, corresponds to force free situation.

So ruling out this condition, the first condition namely $B = 0$, with eqn. (2.79) gives

$$F = \frac{3}{4}\gamma\, A^3 + MA$$

from which $\quad M = \frac{F}{A} - \frac{3}{4}\gamma\, A^2\,.$

Substitution of M yields

$$\cos\omega = \mu/2\,\left(\frac{F}{A} - \frac{3}{4}\gamma\, A^2\right) - \lambda/2\,. \qquad (2.80)$$

Eqn. (2.80) shows the output amplitude variation with the input frequency.

<u>Case 1</u> : If $\gamma = 0.0$ (linear spring)

$$\cos\omega = \frac{F\mu}{2A} - \lambda/2$$

gives the linear system response and is plotted in Fig. 2.5 for various values of the input amplitude F.

<u>Case 2</u> : If $\gamma = 1.0$ (Hard spring)

$$\cos\omega = \frac{\mu}{2}\left[\frac{F}{A} - \frac{3}{4}A^2\right] - \lambda/2\,.$$

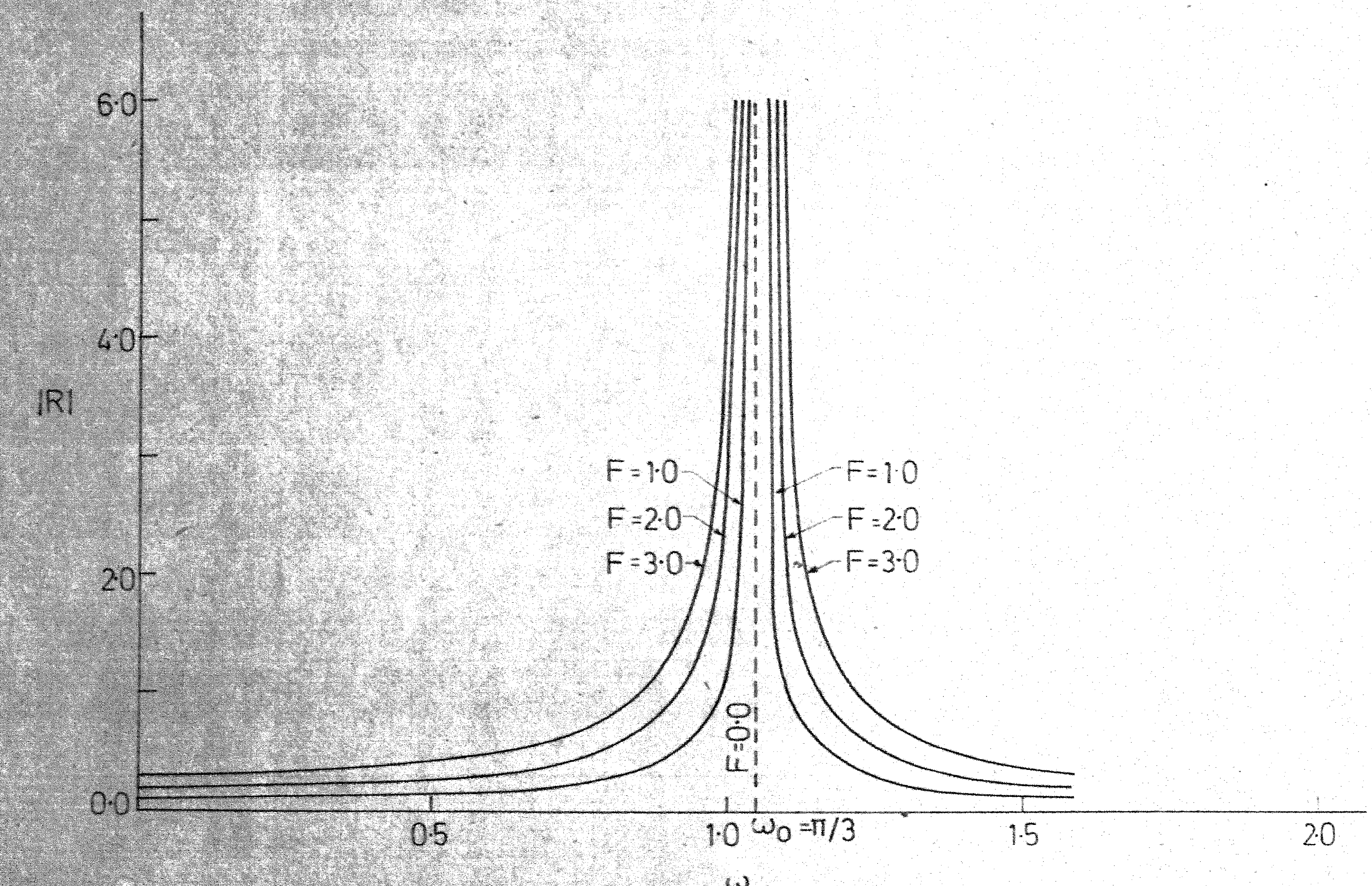


FIG. 2.5 RESPONSE CHARACTERISTICS, $\gamma = 0$
(Linear forced system)

The response characteristics under this situation is shown in Fig. 2.6 for various F. There is a striking resemblance to the familiar response curves of the continuous Duffing equation without damping [5].

<u>Case 3</u> : If $\gamma = -1.0$ (Soft spring)

$$\cos \omega = \frac{\mu}{2}\left[\frac{F}{A} + \frac{3}{4} A^2\right] - \lambda/2.$$

The variation of the amplitude with input frequency for various values of input amplitude is plotted in Fig. 2.7.

In all the three cases the response curve corresponding to $F = 0$ shown by dotted lines is known as 'backbone'. It is interesting to observe that the familiar bending of the response curves for $\gamma \gtrless 0$ about the backbone arises for this discrete time system also.

(b) Duffing equations (with weak damping)

For the case where there is weak viscous damping the discrete model arrived at is

$$x(k) + \lambda\, x(k) + x(k-1) + \mu\, C[x(k+\tfrac{1}{2}) - x(k-\tfrac{1}{2})] + \mu\gamma x^3(k) = \mu[F_1 \cos \omega k - F_2 \sin \omega k], \quad (2.81)$$

where C is the damping coefficient assumed positive.

Adding and subtracting $2 \cos \omega\, x(k)$ on both sides of eqn. (2.81) we obtain

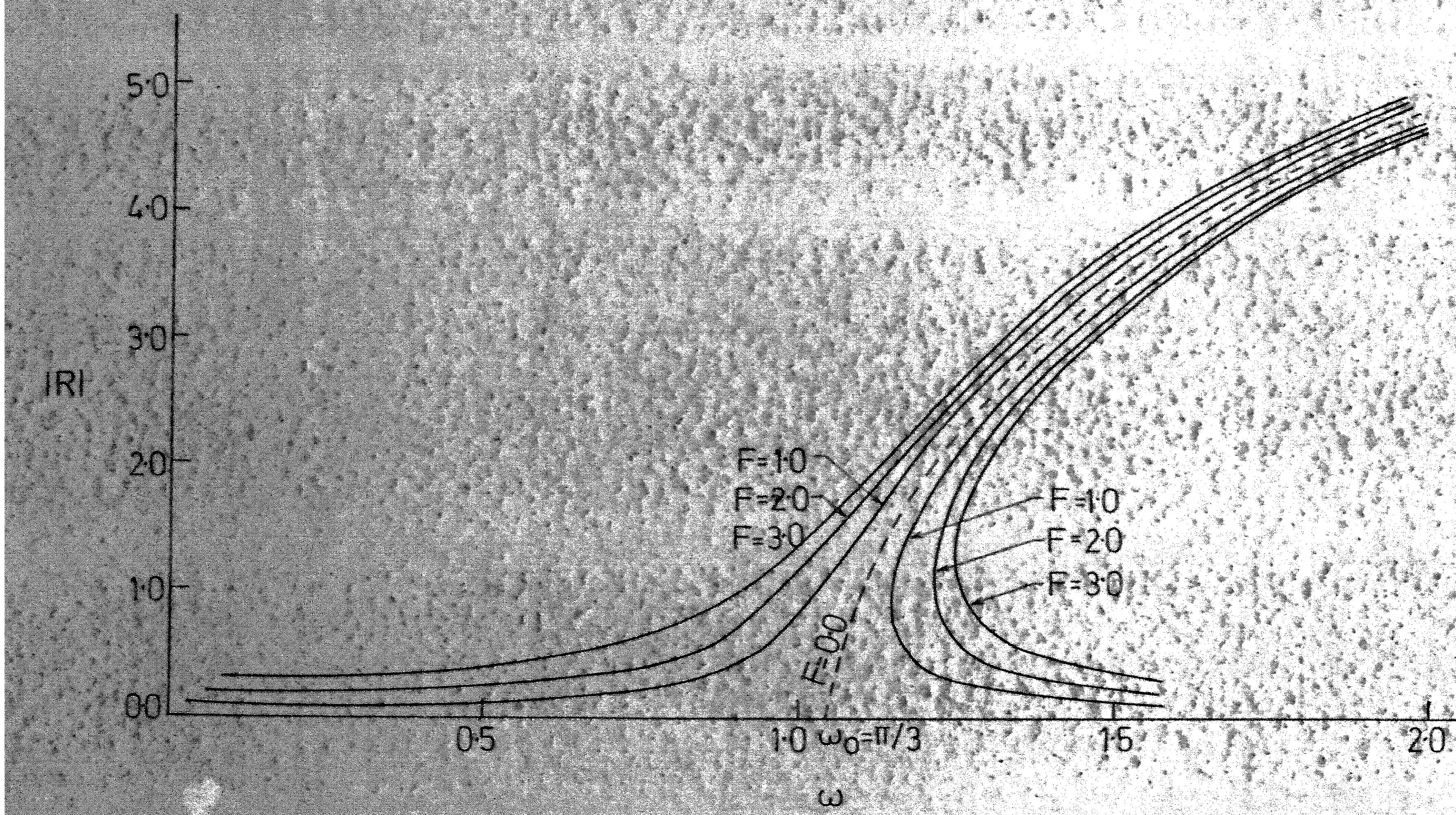


FIG. 2.6 RESRONSE CHARACTERISTICS , γ=1·0

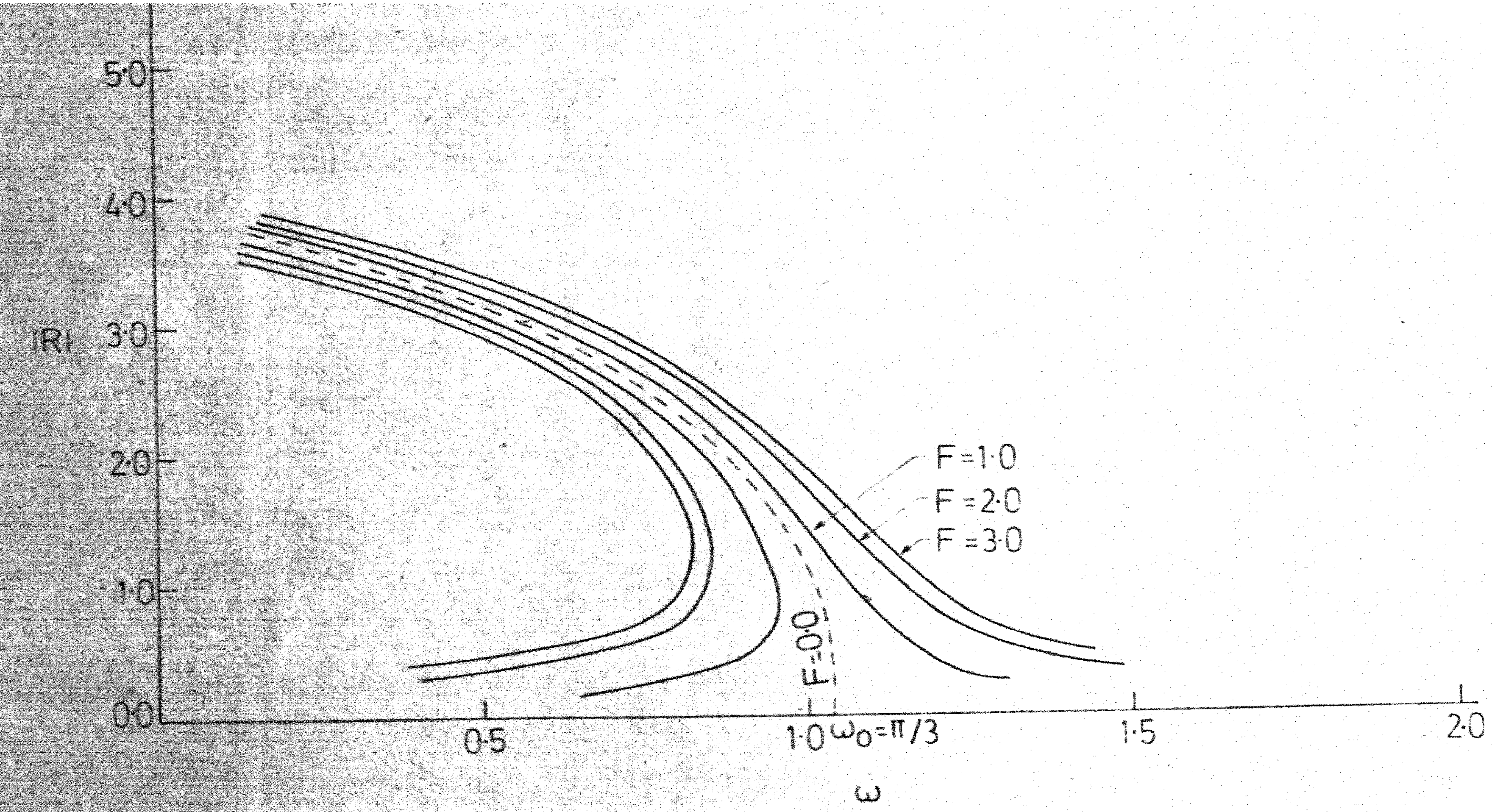


FIG. 2.7 RESPONSE CHARACTERISTICS, $\nu = -1\cdot0$

$$x(k+1) - 2\cos\omega\, x(k) + x(k-1) + \mu M x(k) + \mu\sigma\,[x(k+\tfrac{1}{2}) - x(k-\tfrac{1}{2})]$$
$$+ \mu\gamma x^3(k) = F_1 \cos\omega k - F_2 \sin\omega k . \qquad (2.82)$$

Proceeding as before the basic solution $x_0(\eta,\tau)$ is again obtained as

$$x_0(\eta,\tau) = A(\tau)\cos\omega\eta + B(\tau)\sin\omega\eta .$$

Examining the terms of order μ leads to the following variation for $A(\tau)$ and $B(\tau)$ aftor supression of secular terms

$$\delta A(\tau) = \frac{1}{4\sin\omega/2}\,[F_2 + M B(\tau) - 2\sigma A(\tau)\sin\omega/2 + \tfrac{3}{4}\gamma B(\tau)\,(A^2(\tau) + B^2(\tau)]$$

$$\delta B(\tau) = \frac{1}{4\sin\omega/2}\,[F_1 - M A(\tau) - 2\sigma B(\tau)\sin\omega/2 - \tfrac{3}{4}\gamma A(\tau)\,(A^2(\tau) + B^2(\tau))]$$

from which the steady state behaviour determines

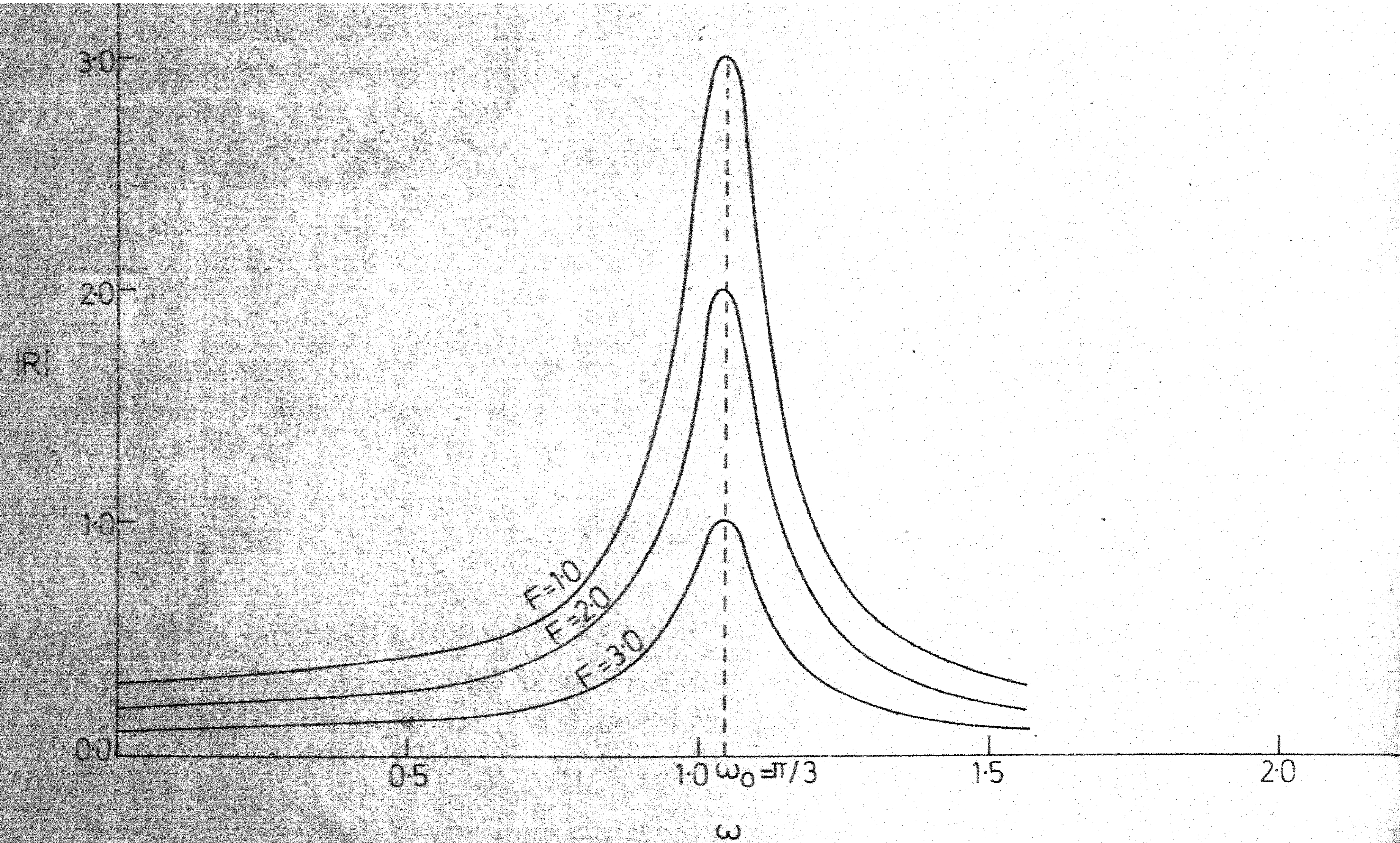
$$F_1 = MA + 2\sigma B\sin\omega/2 + \tfrac{3}{4}\gamma A\,(A^2 + B^2)$$
$$F_2 = -MB + 2\sigma A\sin\omega/2 - \tfrac{3}{4}\gamma B(A^2 + B^2). \qquad (2.83)$$

Let

$$F^2 = F_1^2 + F_2^2$$

where F is the magnitude of the forcing function, that is

$$F^2 = [MA + 2\sigma B\sin\omega/2 + \tfrac{3}{4}\gamma A\,(A^2 + B^2)]^2$$
$$+ [MB - 2\sigma A\sin\omega/2 + \tfrac{3}{4}\gamma B\,(A^2 + B^2)]^2$$
$$= R^2[M^2 + 4\sigma^2\sin^2\omega/2 + \tfrac{9}{16}\gamma^2 R^4 + \tfrac{3}{2}M\gamma R^2] \qquad (2.84)$$



FIG·2.8 RESPONSE CHARACTERISTICS, γ=0, C=1·0
(Linear damped system with forcing function)

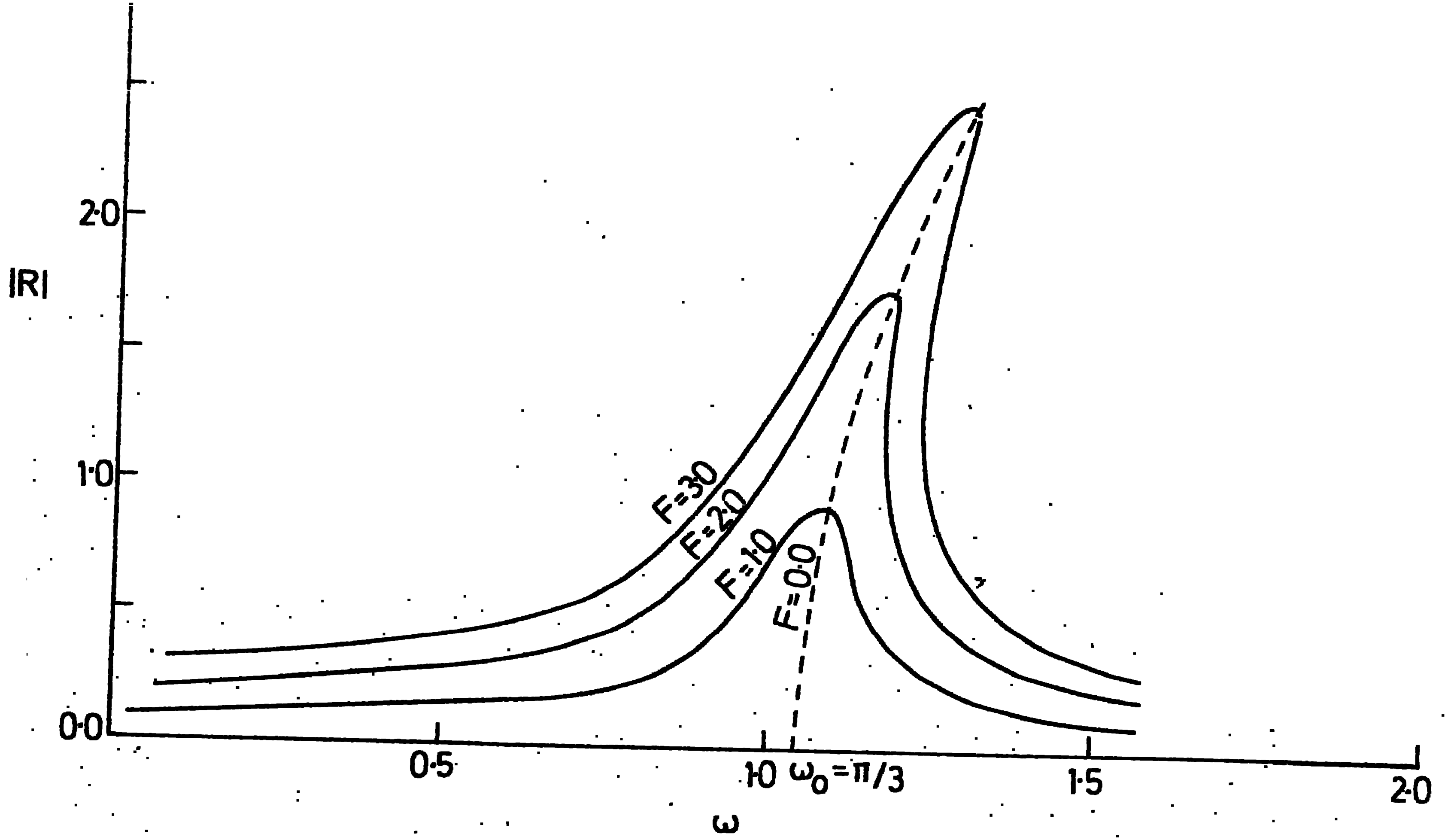


FIG.2.9 RESPONSE CHARACTERISTICS, γ=1·0, C=1·0

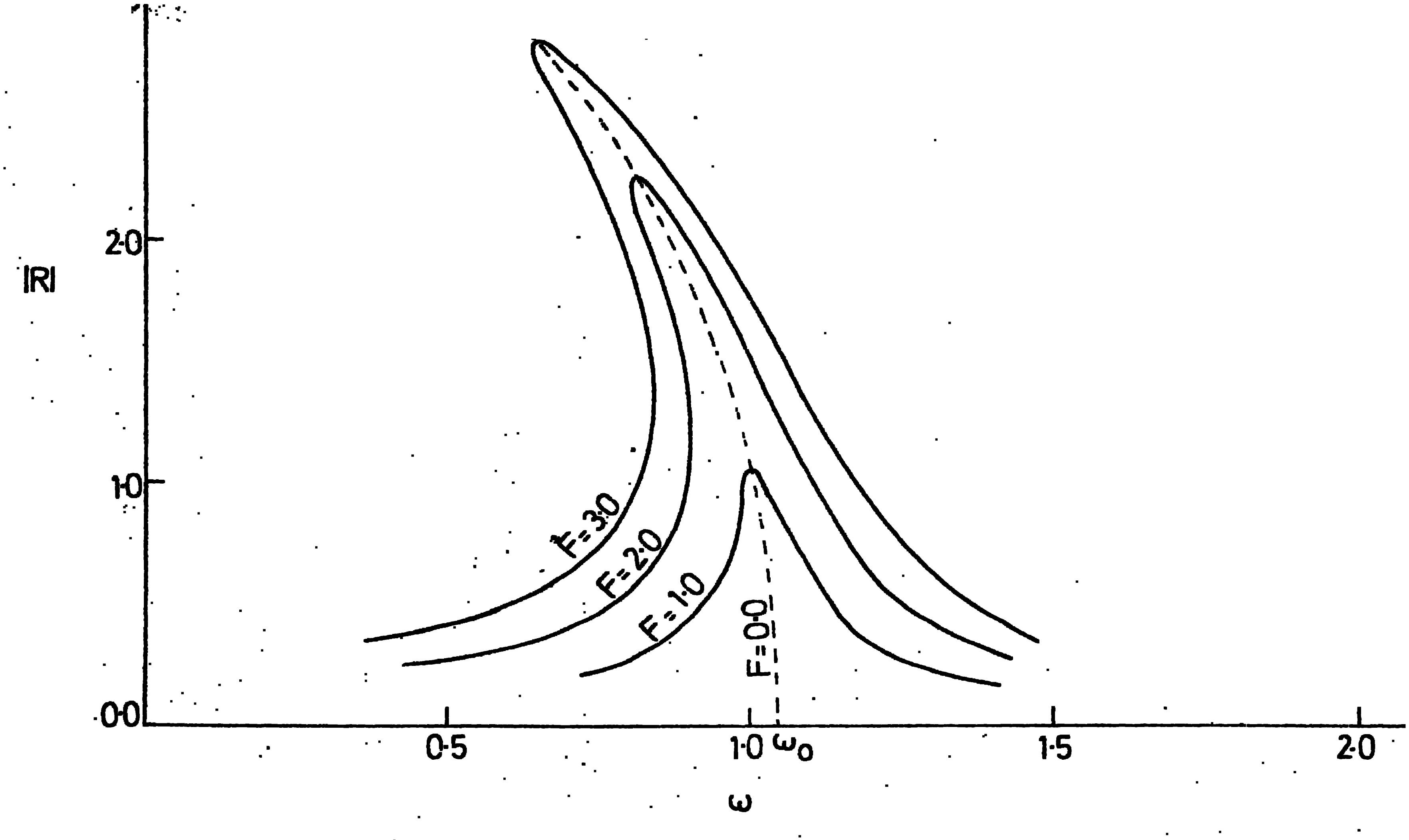


FIG 2.10 RESPONSE CHARACTERISTICS, $\gamma=-1{\cdot}0$, $C=1{\cdot}0$

With $C = 0$ and $F_2 = 0$, the eqn. (2.83) reduces to

$$F_1 = MA + \frac{3}{4}\gamma A(A^2 + B^2)$$
$$0 = MB + \frac{3}{4}\gamma B(A^2 + B^2)$$

which are same as given in eqns. (2.78) and (2.79).

Due to the presence of the weak damping term, closed response curves for the Duffing equation are shown in Figs.. 2.8 to 2.10. It is observed from the figures, for a given amplitude F, each response curve has a vertical tangent as well as horizontal tangent. The equations of the curves describing the locus of the vertical and horizontal tangencies are derived as follows.

Locus of the horizontal tangency points :

Consider the response characteristics given in (2.84), that is

$$F^2 = R^2[M^2 + 4\zeta^2 \sin^2\omega/2 + \frac{9}{16}\gamma^2 R^4 + \frac{3}{2}M\gamma R^2].$$

Here the variables ω and R are assumed to be differentiable with respect to one another.

The locus of the vertical tangency is obtained by imposing the well known condition

$$\frac{dR}{d\omega} = 0 .$$

For convenience the eqn. (2.84) can be rewritten as

$$\frac{F^2}{R^2} = (M + \frac{3}{4}\gamma R^2)^2 + 4\sigma^2 \sin^2 \omega/2 \ .$$

Differentiating with respect to ω , (noting that the amplitude R is a function ω) and setting $\frac{dR}{d\omega} = 0$, we obtain

$$0 = 2[M + \frac{3}{4}\gamma R^2] \frac{dM}{d\omega} + 2\sigma^2 \sin\omega$$

where $M = \frac{2\cos\omega + \lambda}{\mu}$

and $\frac{dM}{d\omega} = -2/\mu \ \sin\omega$.

Substituting in the above equation, we get

$$\cos\omega = \frac{\mu^2 \sigma^2}{4} - \frac{3}{8}\mu\gamma R^2 - \lambda/2 \quad . \tag{2.86}$$

Fig. 2.11 shows a plot of the horizontal loci for the cases $\sigma = 0.0$ and $\sigma = 1.0$, that is without and with weak damping.

Locus of the vertical tangency points :

The locus of the vertical tangents can also be easily obtained from the eqn. (2.84).

The condition to be satisfied here is

$$\frac{d\omega}{dR} = 0.$$

Differentiating the eqn. (2.84) with respect to R and inserting the above condition, we get

$$(M + \frac{3}{4}\gamma R^2)(M + \frac{9}{4}\gamma R^2) + 4\sigma^2 \sin^2\omega/2 = 0 \ .$$

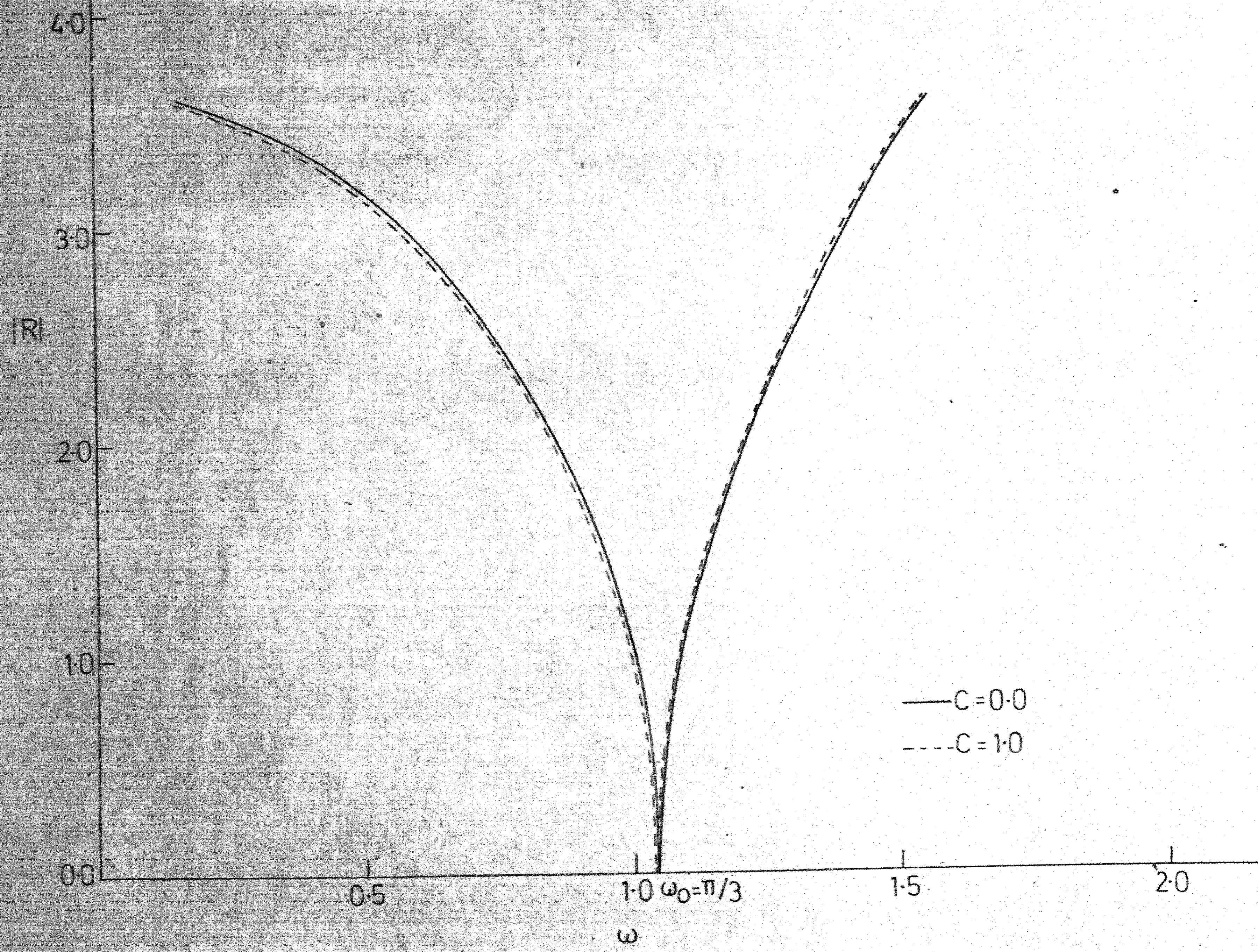


FIG. 2.11 TRACE OF HORIZONTAL TANGENCY

Substituting for M and simplifying, the following equation, representing the locus of the vertical tangents is obtained.

$$\cos^2\omega - P_1 \cos\omega + Q_1 = 0 \qquad (2.87)$$

where

$$P_1 = \frac{\mu^2 c^2}{2} - \frac{3}{2}\mu\gamma R^2 - \lambda$$

$$Q_1 = \frac{1}{4}\left[2\mu^2 c^2 + \frac{27}{16}\mu^2\gamma^2 R^4 + 3\mu\gamma\lambda R^2 + \lambda^2\right].$$

Fig. 2.12 shows a plot of the vertical tangents loci for the cases C = 0.0 and C = 1.0. It is to be noted that for small values of C the vertical tangents loci are close to the curves defined by

$$M + \frac{3}{4}\gamma R^2 = 0$$

$$M + \frac{9}{4}\gamma R^2 = 0.$$

Jump Phenomenon :

Figure 2.13 indicates the amplitude-frequency response characteristics with weak damping for a particular input amplitude F when $\gamma = 1.0$. The 'jump phenomenon' that occurs for variable inputs frequency at a constant amplitude is also shown in the same figure. For increasing ω, 'jump' occurs from a to b and for decreasing ω, 'jump' occurs from c to d. These properties are well known for continuous time systems and the above analysis provides the discrete time counterpart

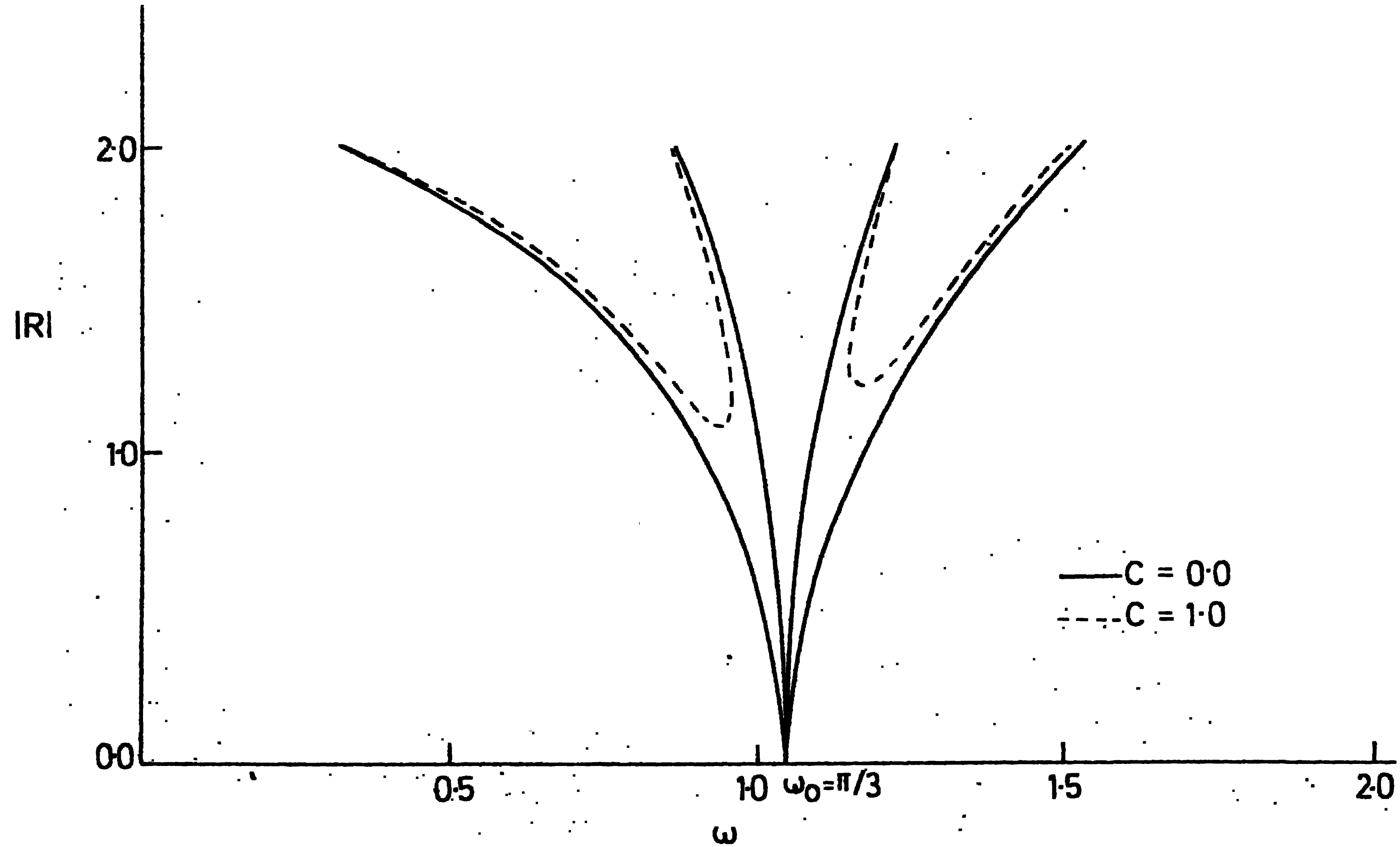


FIG.2.12 TRACE OF VERTICAL TANGENCY

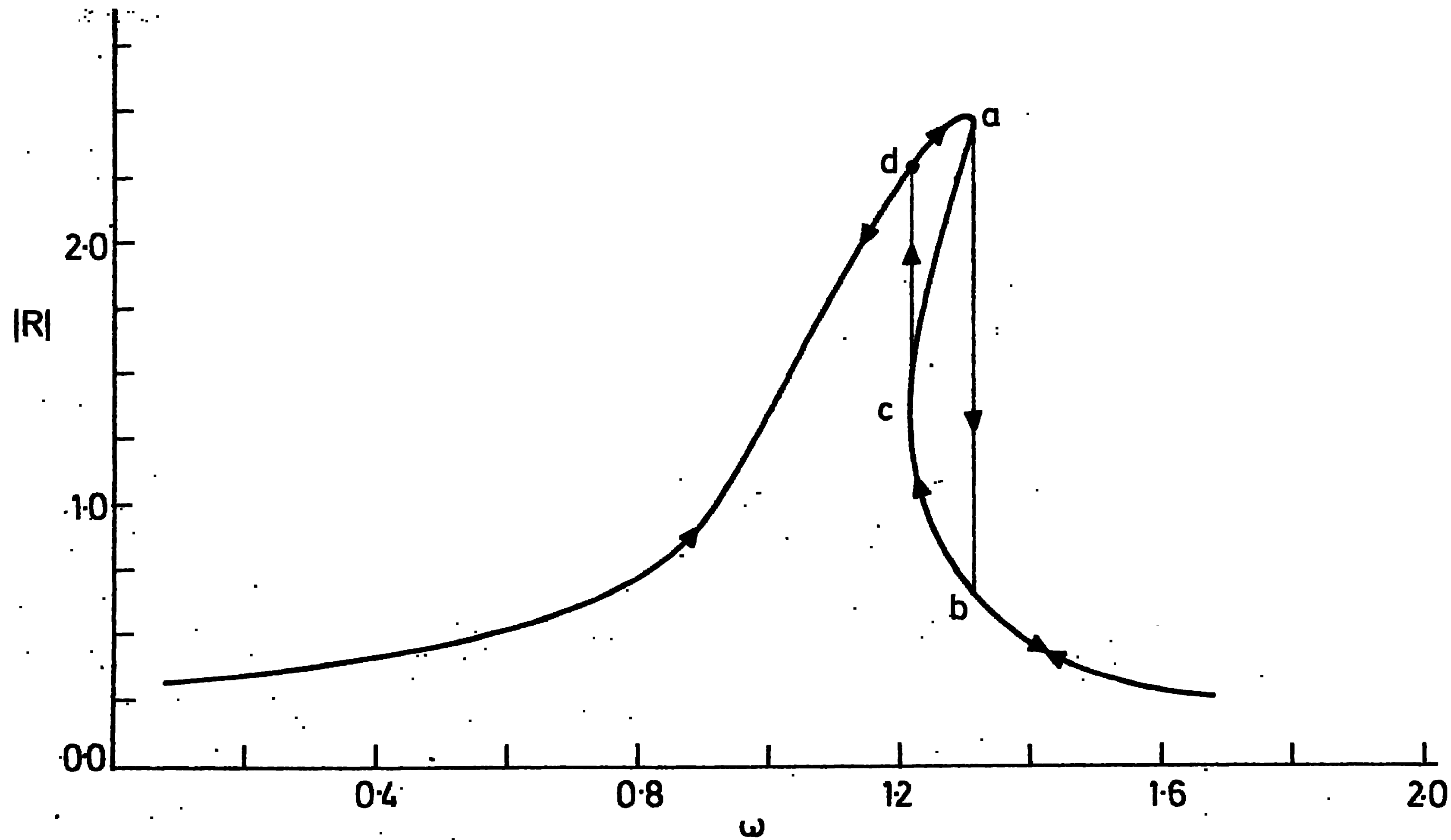


FIG.2.13 JUMP PHENOMENON

as well as supporting the validity of the discrete multiple time perturbation analysis technique. The stability analysis of the output amplitude function can be carried out by locating the singular points in eqn. (2.83) for specified input amplitude and frequency. The above stability study is straight forward and the results are expected to be of the same form as in continuous time systems and hence not considered in the present analysis.

Example 2.7.4 Van der Pol Type Weakly Nonlinear Equation :

A Van der Pol type nonlinear discrete oscillator is considered and its limit cycle behaviour is studied. The system is described by the following difference equation,

$$x(k+1) + x(k-1) + \mu[1 - x^2(k-1)]\, x(k) = 0 . \qquad (2.88)$$

The base solution is

$$x_0(\eta,\tau) = A(\tau)\cos\Theta\eta + B(\tau)\sin\Theta\eta$$

with $\Theta = \pi/2$.

Considering the solution of $x_1(\eta,\tau)$, the following equations for amplitude functions $A(\tau)$ and $B(\tau)$ are obtained.

$$2\sqrt{2}\,\delta B(\tau) = A(\tau)\,[1 - B^2(\tau)]$$

$$2\sqrt{2}\,\delta A(\tau) = -B(\tau)\,[1 - A^2(\tau)] . \qquad (2.89)$$

Under steady state condition

$$A(1 - B^2) = 0$$

$$B(1 - A^2) = 0$$

from which A = 1, B = 1 are the steady state amplitude.

This implies that the Vander Pol type system described in eqn. (2.88) exhibits a limit cycle oscillation of amplitude unity. The stability of such limit cycles are analysed through variational technique as follows :

Before studying the limit cycle stability, it is useful to study the singular points of the system. The real singular point is easily determined to be the origin $x = 0$. The stability of this singular point is now analysed by considering a small variation about it. Let the variation be $x = x^* + u$, where x^* is the singular point. Substituting in the given system the following variational equation is obtained,

$$u(k+1) + u(k-1) - \mu u(k) = 0 .$$

The solution to the above equation is oscillatory, which implies that the singular point is a center type of singularity. This shows the given system exhibits oscillations for the initial conditions which are very close to the origin.

The stability of the limit cycle oscillation of unit magnitude is studied by considering a small perturbation around unity, namely

$$A(\tau) = 1 \pm \partial A(\tau)$$

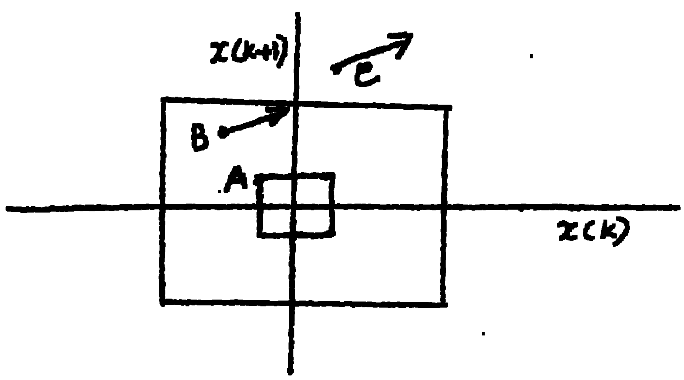
$$B(\tau) = 1 \pm \partial B(\tau) .$$

Substituting these equations in eqn. (2.89) and neglecting higher order terms the following variational equations are obtained :

$$\pm \partial B(\tau+1) \mp \partial B(\tau) \pm \frac{1}{\sqrt{2}} \partial B(\tau) = 0$$

$$\pm \partial A(\tau+1) \mp \partial A(\tau) \mp \frac{1}{\sqrt{2}} \partial A(\tau) = 0 ,$$

It is evident from the above equations $\partial B(\tau) \to 0$ as $\tau \to \infty$ and $\partial A(\tau) \to \infty$ as $\tau \to \infty$, which shows that the limit cycle is of semistable type. This has been verified by simulating the given system equation. The figure given belows illustrates the stability aspects of limit cycle oscillation in the incremental phase plane. In the figure, A is the initial condition



very close to the origin, B and C are the initial conditions inside and outside the periodic limit cycle oscillation (here period 4).

2.8 Modified Definition of Two Time Scaling Method :

As mentioned earlier the proposed scheme works well for all types of equations in a class of discrete time systems. This has been shown in the previous section by considering various kinds of linear as well as nonlinear equations. However, a modified definition over the initially proposed scheme is ~~is~~ introduced here which has a smaller number of mathematical calculations and obtains solutions very close to the exact solutions for certain equations whose solutions are known to be bounded.

The two independent time scales, namely η, the fast time scale and τ, the slow time scale are defined as given in eqn. (2.19) whereas the dependent variable $x(k)$ is expanded in terms of η,τ as shown below :

$$x(k+1) = x(\eta+1,\tau) + \mu[x(\eta+1,\tau+\tfrac{1}{2}) - x(\eta+1,\ \tau-\tfrac{1}{2})]$$

$$x(k-1) = x(\eta-1,\tau) - \mu[x(\eta-1,\tau+\tfrac{1}{2}) - x(\eta-1,\ \tau-\tfrac{1}{2})] \qquad (2.90)$$

The expressions in eqn. (2.90) give the modified definition for the proposed scheme. This modified scheme (scheme 2) is applied to analyse linear as well as a class of nonlinear discrete time systems and the details of the subsequent analysis are exactly similar to those for the proposed scheme (scheme 1).

The results obtained by the schemes 1,2 and 3 are verified with the exact solution of the system equation and a comparative study of these schemes is given in the next section.

2.9 Comparison :

The scheme 2, given in the above section and the scheme 3, proposed in [101] are applied to the examples 2.7.1 and 2.7.2 and the results are compared with the scheme 1 and the exact solution obtained through simulation of the system equation. The plots showing the above comparison are shown in Figs. 2.14 to 2.16. As mentioned earlier the scheme 2 is better than the other schemes when the solution to the system is known to be bounded. The steady state equations obtained for nonlinear systems, by application of all these schemes, are the same and this has been verified by adapting the above schemes to the examples 2.7.3 and 2.7.4.

2.10 Conclusion :

The multiple scale perturbational approach which has been used so far for a class of nonlinear differential and partial differential equations has been successfully employed for analysis of a class of nonlinear discrete time systems described by difference equations. The discrete multiple scale perturbational scheme has been obtained in a systematic way using the known properties of the finite difference operators. Linear and nonlinear examples supporting the usefulness

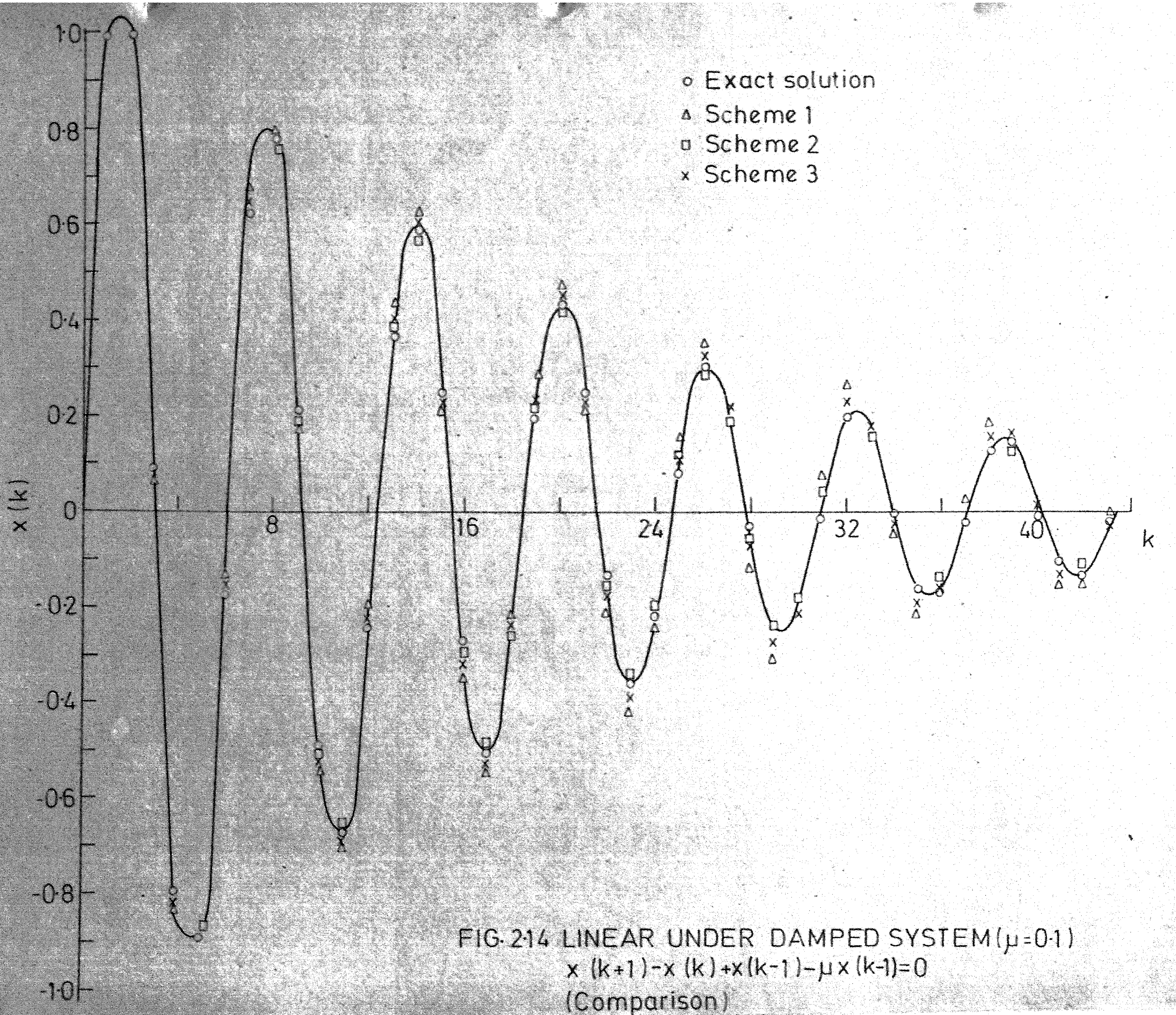


FIG. 2.14 LINEAR UNDER DAMPED SYSTEM ($\mu = 0.1$)

$x(k+1) - x(k) + x(k-1) - \mu x(k-1) = 0$

(Comparison)

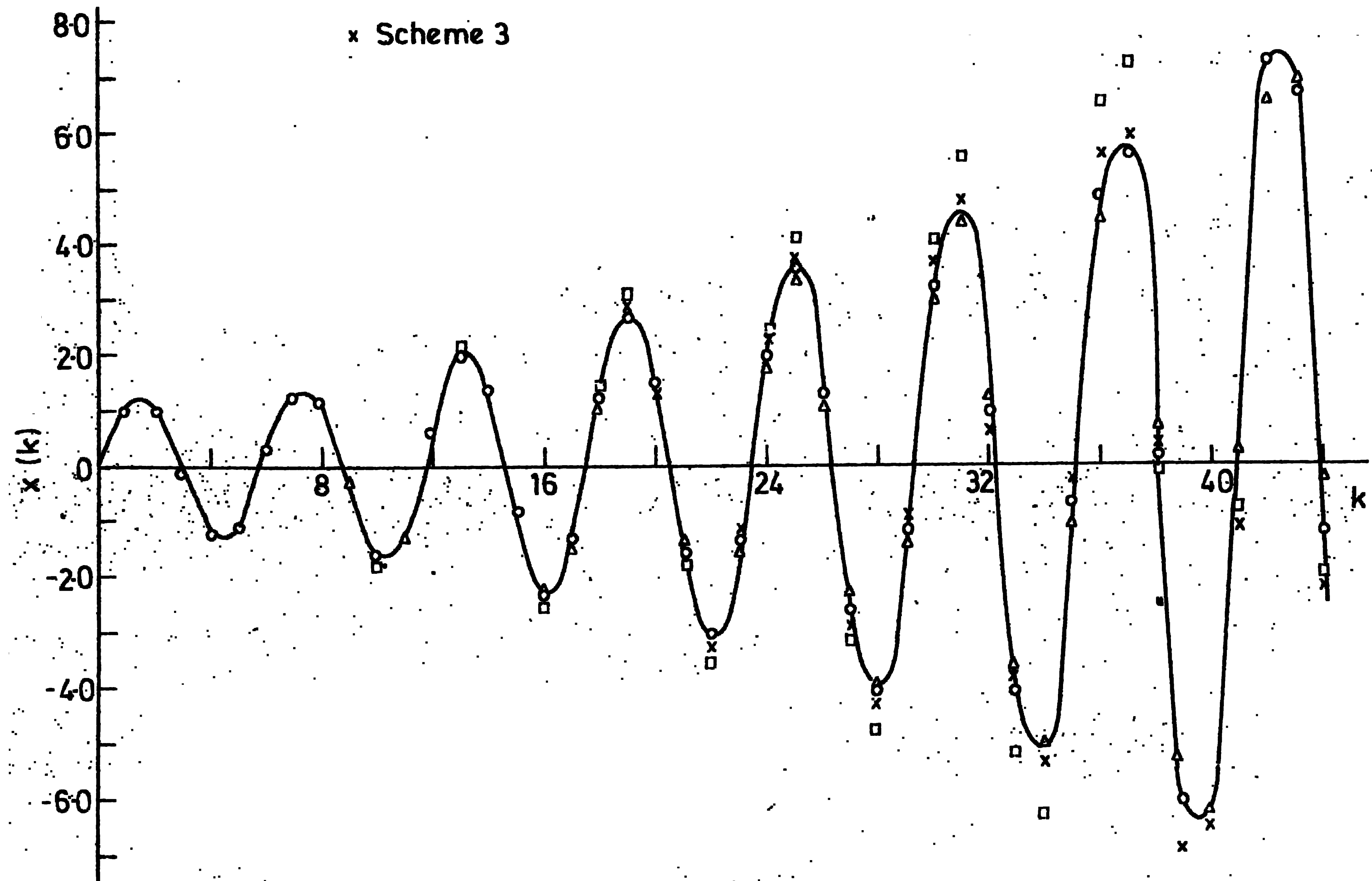
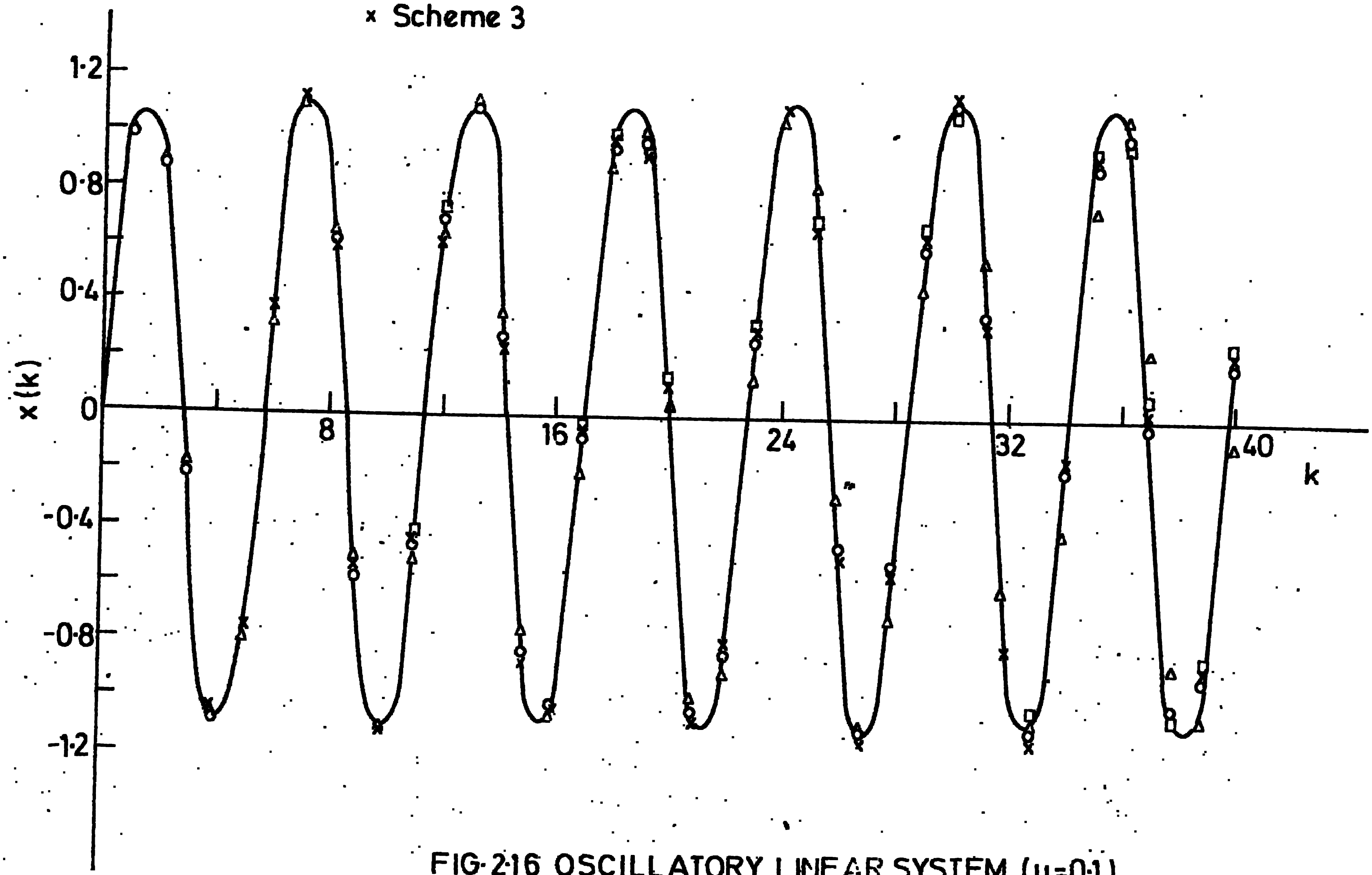


FIG.2.15 UNBOUNDED LINEAR SYSTEM ($\mu = -0.1$)

$x(k+1) - x(k) + x(k-1) - \mu x(k-1) = 0$

(Comparison)



FIG·2·16 OSCILLATORY LINEAR SYSTEM (μ=0·1)

$x(k+1)-x(k)+x(k-1)+\mu x(k)=0$

(Comparison)

of the proposed method have been presented. A modification over the proposed scheme is suggested. This modified scheme involves lesser mathematical computations and is more suitable when the solution to the given nonlinear difference equation is bounded. A comparison of these methods with the scheme 3 as well as with the exact solution of the system equation has also been presented.

CHAPTER 3

STRONGLY FORCED NONLINEAR DIFFERENCE EQUATIONS

3.1 Introduction :

Weakly nonlinear discrete time system under free and weakly forced situations were considered in the previous chapter and several significant results were obtained. In this chapter the study is concentrated on weakly nonlinear discrete time system with strong external input. The study of second order continuous time system under strong forcing has been carried out by many persons. When the forcing is strong, various additional phenomena occur that are not present in weakly forced or in force free systems, specifically, the generation of super/ subharmonic oscillations and frequency entrainment. It is well known that frequency entrainment is possible in a self exited system with external excitation [4,5].

Hayashi [4] has investigated subharmonic solutions for polynomial nonlinearity by the method of harmonic balance. The resulting algebraic equations were solved by iterative method to obtain the required solution. Kronauer and Musa [23] have also reported some general expressions for all subharmonics using multiple time perturbation approach when single input is applied. Tiwari and Subramanian [22] have

recently obtained some necessary conditions for various super, sub and ultrasubharmonic synchronizations in a class of second order differential equation with multiple excitations. Tomas [114] investigated the existence of 3/2 ultrasubharmonics in a Duffing system through the method of harmonic balance. Stanisic et al [115] presented a general perturbation method for the study of sub and super harmonic solutions of weakly nonlinear second order differential equation without damping. Whereas recently Riganti [116] has studied the subharmonic solutions of weakly damped Duffing equation with large nonlinearity.

In this chapter, the method proposed in the previous chapter for analysing the force free and weakly forced second order difference equation, has been adapted to investigate additional phenomena that are generally present in strongly forced situation. Here again the nonlinearity considered for the study is of polynomial type. A Duffing type nonlinear discrete oscillator is considered and various possible super and subharmonic oscillations due to a strong external input are worked out. The well known technique of Harmonic Balance has also been employed to study the above phenomena. The results deduced from the proposed methods are compared with the ones obtained by computer simulation of the system equations.

3.2 System Description

Consider a system which is described by the following difference equation

$$x(k+1) + \alpha x(k) + \beta x(k-1) + \mu f(x(k),\ x(k-1)) = F \cos \omega k, \tag{3.1}$$

where

α and β are the system parameters, μ is the perturbation parameter (generally small) and F, ω are the amplitude and frequency of the input forcing function. The system in eqn. (3.1) is said to be strongly forced if the amplitude F is of $O(1)$.

It is to be noted that, for the location of $[\alpha,\beta]$ in the stability triangle, the linear system response is asymptotically stable, where as the linear response is periodic for $[\alpha,\beta]$ values on the side AC (except at points A and C) of the stability triangle ACG shown in Appendix A. Without loss of generality the value of β can be taken as unity for bounded periodic response with $|\alpha| < 2.0$. A weak damping, to the system described in eqn. (3.1) can be introduced by assuming β nearly equal to unity. With this description the analysis of existence of possible super/sub harmonic solutions is given below.

3.3 Analysis :

Adding and subtracting $x(k-1)$ in eqn. (3.1), it can be rewritten as

$$x(k+1) + \alpha x(k) + x(k-1) + \mu N x(k-1) + \mu f(x(k), x(k-1)$$
$$= F \cos\omega \quad k, \qquad (3.2)$$

where

$\mu N = (\beta-1)$, and N is a damping detuning parameter. It is to be observed that for $\beta = 1$, the detuning parameter $N = 0$ and the given system is free from damping.

Due to forcing, the linear solution, for $\mu = 0$ is obtained as follows :

The linear forced system takes the form

$$x(k+1) + \alpha x(k) + x(k-1) = F \cos\omega \quad k. \qquad (3.3)$$

Let the steady state solution be

$$x(k) = P \cos\omega k + Q \sin_{\omega} t, \qquad (3.4)$$

where P and Q are the constants to be determined from the input periodic function.

Then,

$$x(k+1) = (P \cos\omega + Q \sin\omega) \cos\omega \quad k -$$
$$(P \sin\omega - Q \cos\omega) \sin\omega \quad k$$

$$x(k-1) = (P\cos\omega - Q\sin\omega)\cos\omega k + (P\sin\omega + Q\cos\omega)\sin\omega k,$$

Substituting these relations in eqn. (3.3) and simplifying

$$P(\alpha + 2\cos\omega) = F$$
$$Q(\alpha + 2\cos\omega) = 0, \tag{3.5}$$

the second expression in eqn. (3.5) implies

$$Q = 0 \text{ and } (\alpha + 2\cos\omega) \neq 0.$$

Then the linear forced response is given by

$$x(k) = P\cos\omega k.$$

This response for the linear system suggests the following transformation for the complete nonlinear system (3.2)

$$x(k) = y(k) + P\cos\omega k, \tag{3.6}$$

from which

$$x(k+1) = y(k+1) + P\cos\omega \cos\omega k - P\sin\omega \,.\, \sin\omega k$$
$$x(k-1) = y(k-1) + P\cos\omega \,.\cos\omega k + P\sin\omega \,.\, \sin\omega k,$$

Substitution of these expressions in (3.2) results

$$y(k+1) + \alpha y(k) + y(k-1) + \mu N y(k-1) + \mu f(y(k) + P\cos\omega k,\, y(k-1) + P\cos\omega \,.\, \cos\omega k + P\sin\omega \sin\omega k) = -\mu N (P\cos\omega \,.\, \cos\omega k + P\sin\omega \,.\, \sin\omega k). \tag{3.7}$$

The eqn. (3.7) is a weakly nonlinear discrete time system with weak forcing. Adding and subtracting $2 \cos \frac{n}{m} \omega . y(k)$ in eqn. (3.7) and simplifying,

$$y(k+1) - 2 \cos \frac{n}{m}\omega . y(k) + y(k-1) + \mu\gamma y(k) + \mu N y(k-1)$$

$$+ \mu f(y(k) + P \cos \omega\, k,\ y(k-1) + P \cos\omega \ . \cos\omega \ k$$

$$+ P \sin \omega . \sin\omega k) = -\mu N \ (P \cos_{\omega} . \cos\omega\, k$$

$$+ P \sin\omega \ \sin\omega k), \tag{3.8}$$

where γ is the frequency detuning parameter defined as

$$\gamma\mu = 2 \cos \frac{n}{m}\omega + \alpha, \tag{3.9}$$

and n, m are positive integers.

The solution to eqn. (3.8) can be obtained by applying the discrete multiple time perturbation technique proposed in the previous chapter for a class of discrete time systems. We shall use this technique and will represent the solution of (3.8) by an asymptotic expansion involving two time variables ;

$$y(\eta,\ \tau;\ \mu) = y_0(\eta,\tau) + \mu\ y_1(\eta,\tau) + \mu^2 y_2(\eta,\tau) + \ldots \tag{3.10}$$

where η is the fast time and τ is the slow time and are treated as independent. The definition of these independent scales and the expansion of the variables x(k), x(k+1) and x(k-1) in terms of these independent variables are given in the previous chapter. It is to be noted that in the first order approximation to the solution the independent scales assume the

following simple form

$$\eta = k$$
$$\tau = \mu k.$$

Then

$$y(k) = y(\eta,\tau)$$

$$y(k+1) + y(k-1) = y(\eta+1,\tau) + y(\eta-1,\tau) + 2\mu[y(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - y(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - y(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + y(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})]. \quad (3.11)$$

Substituting the eqn. (3.11) in eqn. (3.8)

$$y(\eta+1,\tau) - 2\cos\frac{n}{m}\omega\, y(\eta,\tau) + y(\eta-1,\tau) + 2\mu[y(n+\tfrac{1}{2},\tau+\tfrac{1}{2}) - y(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - y(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + y(\eta-\tfrac{1}{2},\tau-\tfrac{1}{2})] + \mu\gamma y(\eta,\tau) + \mu N y(\eta-1,\tau) + \mu f[y(\eta,\tau) + P\cos^{\omega}\ \eta,\ y(\eta-1,\tau) + \mu(\cdot) + P\cdot\cos\omega\ \cos\omega\eta + P\sin\omega\sin\omega\ \eta] = - \mu N(P\cos\omega.\ \cos\omega\ \eta + P\sin\omega\ .\ \sin\omega\ \eta). \quad (3.12)$$

Combining the eqns. (3.10) and (3.12) and collecting terms of like order in μ, the following equations are obtained

μ^{o} terms : $y_o(\eta+1,\tau) - 2\cos\frac{n}{m}\omega\, y_o(\eta,\tau) + y_o(\eta-1,\tau) = 0$ (3.13)

μ terms : $y_1(\eta+1,\tau) - 2\cos\frac{n}{m}\omega\ y_1(\eta,\tau) + y_1(\eta-1,\tau) =$

$$-N(P\cos\omega\ .\ \cos^{\omega}\eta + P\sin\omega.\ \sin^{\omega}\eta) - f(y_o(\eta,\tau) + P\cos\omega\eta,\ y_o(\eta-1,\tau) + P\cos^{\omega}\ .\ \cos\omega\eta + P\sin\omega\ .\ \sin\omega\ \eta) - N\, y_o(\eta-1,\tau) - \gamma y_o(\eta,\tau) - 2[y_o(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) - y_o(\eta+\tfrac{1}{2},\tau-\tfrac{1}{2}) - y_o(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) + y_o(\eta-\tfrac{1}{2},\ \tau-\tfrac{1}{2})]. \quad (3.14)$$

The solution to eqn. (3.13) is known as basic or generating solution and is given by

$$y_o(\eta,\tau) = A(\tau)\cos\frac{n}{m}\omega\eta + B(\tau)\sin\frac{n}{m}\omega\eta. \qquad (3.15)$$

Note that the response $y_o(\eta,\tau)$ is purely oscillatory and this arises primarily because of the particular from of $\mu\gamma$ that is defined in (3.9).

Where

A and B represent the amplitude functions of

i) Subharmonics, if $n/m < 1.0$ for $n = 1$ and $m \geq 2$

ii) Superharmonic, if $n/m > 1.0$ for $m = 1$ and $n \geq 2$ and

iii) Ultrasubharmonics, if $n/m \lessgtr 1.0$ for n and m relative prime integers.

The solution in (3.15) is then substituted into the right hand side of eqn. (3.14) and the secular terms are suppressed as follows.

Let $\omega_1 = \frac{n}{m}\omega$, then

$$y_o(\eta+\tfrac{1}{2},\tau+\tfrac{1}{2}) = [A(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2} + B(\tau+\tfrac{1}{2})\sin\frac{\omega_1}{2}]\cos\omega_1\eta$$

$$- [A(\tau+\tfrac{1}{2})\sin\frac{\omega_1}{2} - B(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2}]\sin\omega_1\eta$$

$$y_o(\eta-\tfrac{1}{2},\tau+\tfrac{1}{2}) = [A(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2} - B(\tau+\tfrac{1}{2})\sin\frac{\omega_1}{2}]\cos\omega_1\eta$$

$$+ [A(\tau+\tfrac{1}{2})\sin\frac{\omega_1}{2} + B(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2}]\sin\omega_1\eta$$

the right hand side of eqn. (3.14) becomes

$$\text{RHS} = -N(P\cos\omega\ \cos\omega\eta + P\sin\omega\ ,\ \sin\omega\eta) - a_1\cos\omega_1\eta$$
$$- b_1\sin\omega_1\eta - N[A(\tau)\cos\omega_1 - B(\tau)\sin\omega_1]\cos\omega_1\eta$$
$$- N[A(\tau)\sin\omega_1 + B(\tau)\cos\omega_1]\sin\omega_1\eta - \gamma A(\tau)\cos\omega_1\eta$$
$$\gamma B(\tau)\sin\omega_1\eta - 2[A(\tau+\tfrac{1}{2})\cos\omega_1/2 + B(\tau+\tfrac{1}{2})\sin^{\omega}/2]$$
$$\cos\omega_1\eta + 2\left[A(\tau+\tfrac{1}{2})\sin\frac{\omega_1}{2} - B(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2}\right]\sin\omega_1\eta$$
$$+ 2\left[A(\tau-\tfrac{1}{2})\cos\frac{\omega_1}{2} + B(\tau-\tfrac{1}{2})\sin\frac{\omega_1}{2}\right]\cos\omega_1\eta$$
$$- 2\left[A(\tau-\tfrac{1}{2})\sin\frac{\omega_1}{2} - B(\tau-\tfrac{1}{2})\cos\frac{\omega_1}{2}\right]\sin\omega_1\eta$$
$$+ 2\left[A(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2} - B(\tau+\tfrac{1}{2})\sin\frac{\omega_1}{2}\right]\cos\omega_1\eta + 2[A(\tau+\tfrac{1}{2})$$
$$\sin\frac{\omega_1}{2} + B(\tau+\tfrac{1}{2})\cos\frac{\omega_1}{2}]\sin\omega_1\eta - 2[A(\tau-\tfrac{1}{2})\cos\frac{\omega_1}{2}$$
$$- B(\tau-\tfrac{1}{2})\sin\frac{\omega_1}{2}]\cos\ _1\eta - 2\left[A(\tau-\tfrac{1}{2})\sin\frac{\omega_1}{2} + B(\tau-\tfrac{1}{2})\cos\frac{\omega_1}{2}\right]$$
$$\sin\omega_1\eta.$$

Equating the coefficients of $\cos\omega_1\eta$ and $\sin\omega_1\eta$ terms separately to zero (secular term elimination)

$\cos\omega_1\eta$ terms :

$$-a_1 - N(A(\tau)\cos\omega_1 - B(\tau)\sin\omega_1) - \gamma A(\tau) - 4\delta B(\tau)\sin\frac{\omega_1}{2} = 0$$

that is,

$$4\sin\frac{\omega_1}{2}\,\delta B(\tau) = -a_1 - N(A(\tau)\cos\omega_1 - B(\tau)\sin\omega_1) - \gamma A(\tau) \tag{3.16}$$

$\sin\omega_1\eta$ terms :

$$-b_1 - N(A(\tau)\sin\omega_1 + B(\tau)\cos\omega_1) - \gamma B(\tau) + 4\delta A(\tau)\sin\frac{\omega_1}{2} = 0$$

that is

$$4\sin\frac{\omega_1}{2}\,\delta A(\tau) = b_1 + N\,(A(\tau)\sin\omega_1 + B(\tau)\cos\omega_1) + \gamma B(\tau). \tag{3.17}$$

Eqns. (3.16) and (3.17) are first order coupled difference equations (usually nonlinear) in $A(\tau)$ and $B(\tau)$. In eqns. (3.16) and (3.17), a_1 and b_1 are the coefficients of the fundamental components $\cos\omega_1\eta$ and $\sin\omega_1\eta$ of the nonlinear function f in eqn. (3.14) and δ is the central difference operator defined as

$$\delta A(\tau) = A(\tau + \tfrac{1}{2}) - A(\tau - \tfrac{1}{2})$$

and likewise $\delta B(\tau)$.

Under steady state condition

$$\delta A(\tau) = \delta(B)(\tau) = 0 \text{ and } A(\tau) = A,\ B(\tau) = B$$

where A and B are the steady state values of $A(\tau)$ and $B(\tau)$ respectively. Then the equations (3.16) and (3.17) take the form

$$\begin{aligned} a_1 + N(A\cos\omega_1 - B\sin\omega_1) + \gamma A &= 0 \\ b_1 + N(A\sin\omega_1 + B\cos\omega_1) + \gamma B &= 0 \end{aligned} \tag{3.18}$$

Eqn. (3.18) gives the general steady state condition on the amplitude functions $A(\tau)$ and $B(\tau)$ for a general nonlinear function f.

In the next section a particular form for the nonlinear function f is assumed and the possible super/sub harmonic solutions are investigated.

3.4 Example :

Consider the following nonlinear difference equation with weak cubic nonlinearity. This discrete oscillator may be considered as a Duffing type with weak damping.

$$x(k+1) - x(k) + \beta x(k-1) + \mu x^3(k) = F \cos \omega k \qquad (3.19)$$

Comparing (3.19) with (3.1)

$$\alpha = -1.0$$
$$f(\cdot) = x^3(k).$$

It is to be noted that as stated earlier in this chapter, β may be taken to be slightly less than unity to introduce small damping in the system.

The nonlinearity given in (3.19) takes the following form due to the transformation of variable [eqn. (3.6)].

$$\begin{aligned} x^3(k) &= [y(k) + P \cos \omega k]^3 \\ &= y^3(k) + 3Py^2(k) \cos\omega k + 3P^2 y(k) \cos^2 \omega k \\ &\qquad + P^3 \cos^3 \omega k. \end{aligned}$$

Due to the introduction of two independent time scales the above expression can be rewritten in the following form

$$x^3(\eta,\tau) = y^3(\eta,\tau) + 3Py^2(\eta,\tau)\cos\omega\,\eta + 3P^2y(\eta,\tau)\cos^2\omega\,\eta + P^3\cos^3\omega\,\eta.$$

Then using eqn. (3.10) the above expression can be rewritten as

$$x^3(\eta,\tau) = y_o^3(\eta,\tau) + 3Py_o^2(\eta,\tau)\cos\omega\,\eta + 3P^2y_o(\eta,\tau)\cos^2\omega\,\eta + P^3\cos^3\omega\,\eta + O(\mu).$$

Substituting for $y_o(\eta,\tau)$ from eqn. (3.15)

$$\begin{aligned} y_o^3(\eta,\tau) &+ 3P^2y_o(\eta,\tau)\cos\omega\,\eta + 3P^2y_o(\eta,\tau)\cos^2\omega\,\eta + P^3\cos^3\omega\,\eta \\ &= [A(\tau)\cos\omega_1\eta + B(\tau)\sin\omega_1\eta]^3 + 3P\cos\omega\,\eta[A(\tau)\cos\omega_1\eta \\ &+ B(\tau)\sin\omega_1\eta]^2 + 3P^2\cos^2\omega\,\eta[A(\tau)\cos\omega_1\eta \\ &+ B(\tau)\sin\omega_1\eta] + P^3\cos^3\omega\,\eta. \end{aligned} \tag{3.20}$$

$$\begin{aligned} \text{RHS of (3.20)} &= A^3(\tau)\cos^3\omega_1\eta + 3A^2(\tau)B(\tau)\cos^2\omega_1\eta\sin\omega_1\eta \\ &+ 3A(\tau)B^2(\tau)\cos\omega_1\eta\sin^2\omega_1\eta + B^3(\tau)\sin^3\omega_1\eta \\ &+ 3P\cos\omega\,\eta[A^2(\tau)\cos^2\omega_1\eta + 2A(\tau)B(\tau)\cos\omega_1\eta. \\ &\sin\omega_1\eta + B^2(\tau)\sin^2\omega_1\eta] + 3P^2\cos^2\omega\,\eta[A(\tau)\cos\omega_1\eta \\ &+ B(\tau)\sin\omega_1\eta] + P^3\cos^3\omega\,\eta. \end{aligned}$$

Using well known trigonometric identities, the above expression can be rewritten as

$$= [\tfrac{3}{4}A^3(\tau) + \tfrac{3}{4}A(\tau)B^2(\tau) + \tfrac{3}{2}P^2A(\tau)] \cos \omega_1\eta + [\tfrac{3}{4}B^3(\tau) + \tfrac{3}{4}A^2(\tau)B(\tau) + \tfrac{3}{2}P^2B(\tau)] \sin \omega_1\eta + [\tfrac{1}{4}A^3(\tau) - \tfrac{3}{4}A(\tau)B^2(\tau)] \cos 3\,\omega_1\eta - [\tfrac{1}{4}B^3(\tau) - \tfrac{3}{4}A^2(\tau)B(\tau)] \sin 3\,\omega_1\eta + [\tfrac{3}{2}PA^2(\tau) + \tfrac{3}{2}PB^2(\tau) + \tfrac{3}{4}P^3] \cos\omega\,\eta + \tfrac{1}{4}P^3\cos 3\omega\,\eta + [\tfrac{3}{4}PA^2(\tau) - \tfrac{3}{4}PB^2(\tau)] \cos(\omega + 2\,\omega_1)\eta + [\tfrac{3}{4}PA^2(\tau) - \tfrac{3}{4}PB^2(\tau)] \cos(\omega - 2\,\omega_1)\eta + \tfrac{3}{2}PA(\tau)B(\tau) \sin(\omega + 2\,\omega_1)\eta + \tfrac{3}{2}PA(\tau)B(\tau) \sin(2\,\omega_1 - \omega)\eta + \tfrac{3}{4}P^2A(\tau) \cos(\omega_1 + 2\omega)\eta + \tfrac{3}{4}P^2A(\tau) \cos(\omega_1 - 2\omega)\eta + \tfrac{3}{4}P^2B(\tau) \sin(\omega_1 + 2\omega)\eta + \tfrac{3}{4}P^2B(\tau) \sin(\omega_1 - 2\omega)\eta. \qquad (3.21)$$

From the above relation the coefficients a_1 and b_1 (introduced in eqns. (3.16) and (3.17)) of $\cos \omega_1\eta$ and $\sin \omega_1\eta$ terms can be determined for various orders of subharmonic oscillations.

(i) Subharmonic of order 2 :

A pth harmonic component has a period $1/p$ of the period of the fundamental component. When $p = \frac{1}{q}$, where q is an integer, the order of harmonic is $1/q$. Such a fractional order of harmonic is called a subharmonic of order q.

Consider a subharmonic of order 2, the period of which is twice the period of the fundamental component.

Here $q = 2$, that is

$$\omega_1 = \omega/q = \omega/2.$$

Substituting for ω_1 in the eqn. (3.21) and collecting the coefficients of $\cos \omega/2\, \eta$ and $\sin \omega/2\, \eta$ terms, we have

$$\begin{aligned} a_1 &= \tfrac{3}{4}A^3 + \tfrac{3}{4}AB^2 + \tfrac{3}{2}P^2A \\ &= \tfrac{3}{4}AR^2 + \tfrac{3}{2}P^2A \\ b_1 &= \tfrac{3}{4}B^3 + \tfrac{3}{4}A^2B + \tfrac{3}{2}P^2A \\ &= \tfrac{3}{4}BR^2 + \tfrac{3}{2}P^2B, \end{aligned} \qquad (3.22)$$

where

$$R^2 = A^2 + B^2.$$

Now the eqn. (3.18) becomes

$$A(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N \cos \omega_1 + \gamma) - NB \sin \omega_1 = 0$$

$$B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N \cos \omega_1 + \gamma) + NA \sin \omega_1 = 0, \qquad (3.23)$$

multiplying the first equation by B and the second by A and subtracting

$$R^2N \sin \omega_1 = 0,$$

which implies

$$R^2 = 0 \qquad \text{for } N \neq 0,$$

that is the amplitude of subharmonic oscillation of order 2 is zero as long as damping is present. Therefore subharmonic

oscillation of order 2 can not occur in this case. On the other hand for $N = 0$, that is for $\beta = 1.0$ (zero damping), we have

$$R^2(\tfrac{3}{4}R^2 + \gamma + \tfrac{3}{2}P^2) = 0,$$

from which

$$R^2 = 0 \quad \text{or} \quad \tfrac{3}{4}R^2 + \gamma + \tfrac{3}{2}P^2 = 0.$$

Ruling out the first solution (which leads to trivial solution) the require condition is

$$\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + \gamma = 0. \tag{3.24}$$

The eqn. (3.24) gives the relationship between the amplitude of subharmonic of order 2 and the excitation frequency ω.

Again from the eqn. (3.24)

$$R^2 = -\tfrac{4}{3}\,[\tfrac{3}{2}P^2 + \gamma] \tag{3.25}$$

(ii) Subharmonic of order 3 :

For this case $\omega_1 = \omega/3$.

Substituting $\omega_1 = \omega/3$ in (3.21) and collecting the coefficients of cos $\omega/3\ \eta$ and sin $\omega/3\ \eta$, we obtain

$$\begin{aligned} a_1 &= \tfrac{3}{4}A^3 + \tfrac{3}{4}AB^2 + \tfrac{3}{2}P^2A + \tfrac{3}{4}PA^2 - \tfrac{3}{4}PB^2 \\ &= \tfrac{3}{4}AR^2 + \tfrac{3}{2}AP^2 + \tfrac{3}{4}P(A^2 - B^2) \end{aligned}$$

$$b_1 = \frac{3}{4}BR^2 + \frac{3}{2}BP^2 - \frac{3}{2}PAB,$$

with these values of a_1 and b_1, the eqn. (3.18) becomes

$$A[\frac{3}{4}R^2 + \frac{3}{2}P^2 + N\cos\omega_1 + \gamma] + \frac{3}{4}P(A^2 - B^2) - NB\sin\omega_1 = 0 \tag{3.26}$$

$$B[\frac{3}{4}R^2 + \frac{3}{2}P^2 + N\cos\omega_1 + \gamma] - \frac{3}{2}PAB + NA\sin\omega_1 = 0. \tag{3.27}$$

Let $S = \frac{3}{4}R^2 + \frac{3}{2}P^2 + N\cos\omega_1 + \gamma$

Then the eqns. (3.26) and (3.27) become

$$AS - NB\sin\omega_1 = -\frac{3}{4}P(A^2 - B^2)$$

$$BS + NA\sin\omega_1 = \frac{3}{2}PAB.$$

Squaring and adding

$$S^2 + N^2\sin^2\omega_1 = \frac{9}{16}P^2R^2. \tag{3.28}$$

Substituting for S and simplifying, we have

$$\frac{9}{16}R_1^2 + [\frac{27}{16}P^2 + \frac{3}{2}(N\cos\omega_1 + \gamma)]R_1 + [\frac{9}{4}P^4 + 3\gamma P^2 + \gamma^2$$
$$+ N(N + \cos\omega_1(2\gamma + 3P^2))] = C, \tag{3.29}$$

where

$$R_1 = R^2$$

$$P = F/(2\cos\omega + \alpha).$$

For $N = 0$, that is for dissipation free system

$$\frac{9}{16}R_1^2 + (\frac{27}{16}P^2 + \frac{3}{2}\gamma)R_1 + (\frac{3}{2}P^2 + \gamma)^2 = 0. \tag{3.30}$$

(iii) Subharmonic of order 4 :

$$\omega_1 = \omega/q = \omega/4.$$

Then from the eqn. (3.21), we have

$$a_1 = \tfrac{3}{4}A^2 + \tfrac{3}{4}AB^2 + \tfrac{3}{2}PA^2 = \tfrac{3}{4}AR^2 + \tfrac{3}{2}AP^2$$

$$b_1 = \tfrac{3}{4}B^3 + \tfrac{3}{4}A^2B + \tfrac{3}{2}P^2B = \tfrac{3}{4}BR^2 + \tfrac{3}{2}BP^2.$$

Combining the values of a_1 and b_1 with eqn. (3.18)

$$\tfrac{3}{4}AR^2 + \tfrac{3}{2}AP^2 + N(A\cos\omega_1 - B\sin\omega_1) + \gamma A = 0$$

$$\tfrac{3}{4}BR^2 + \tfrac{3}{2}P^2A + N(A\sin\omega_1 + B\cos\omega_1) + \gamma B = 0,$$

that is

$$A[\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N\cos\omega_1 + \gamma] - NB\sin\omega_1 = 0 \qquad (3.31)$$

$$B[\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N\cos\omega_1 + \gamma] + NA\sin\omega_1 = 0. \qquad (3.32)$$

Multiplying (3.31) by B and (3.32) by A and subtracting.

$$R^2N\sin\omega_1 = 0,$$

which implies that the amplitude of 4th order subharmonic oscillation is zero whenever there is damping in the system. This shows the subharmonic oscillation of order 4 cannot occur in this case. However, with zero damping, the eqns. (3.31) and (3.32) reduce to

$$\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + \gamma = 0, \tag{3.33}$$

where $\mu\gamma = 2\cos\omega/4 + \alpha$.

(iv) Subharmonic of order 5 :

With $\omega_1 = \omega/5$, following the steps given above, we have,

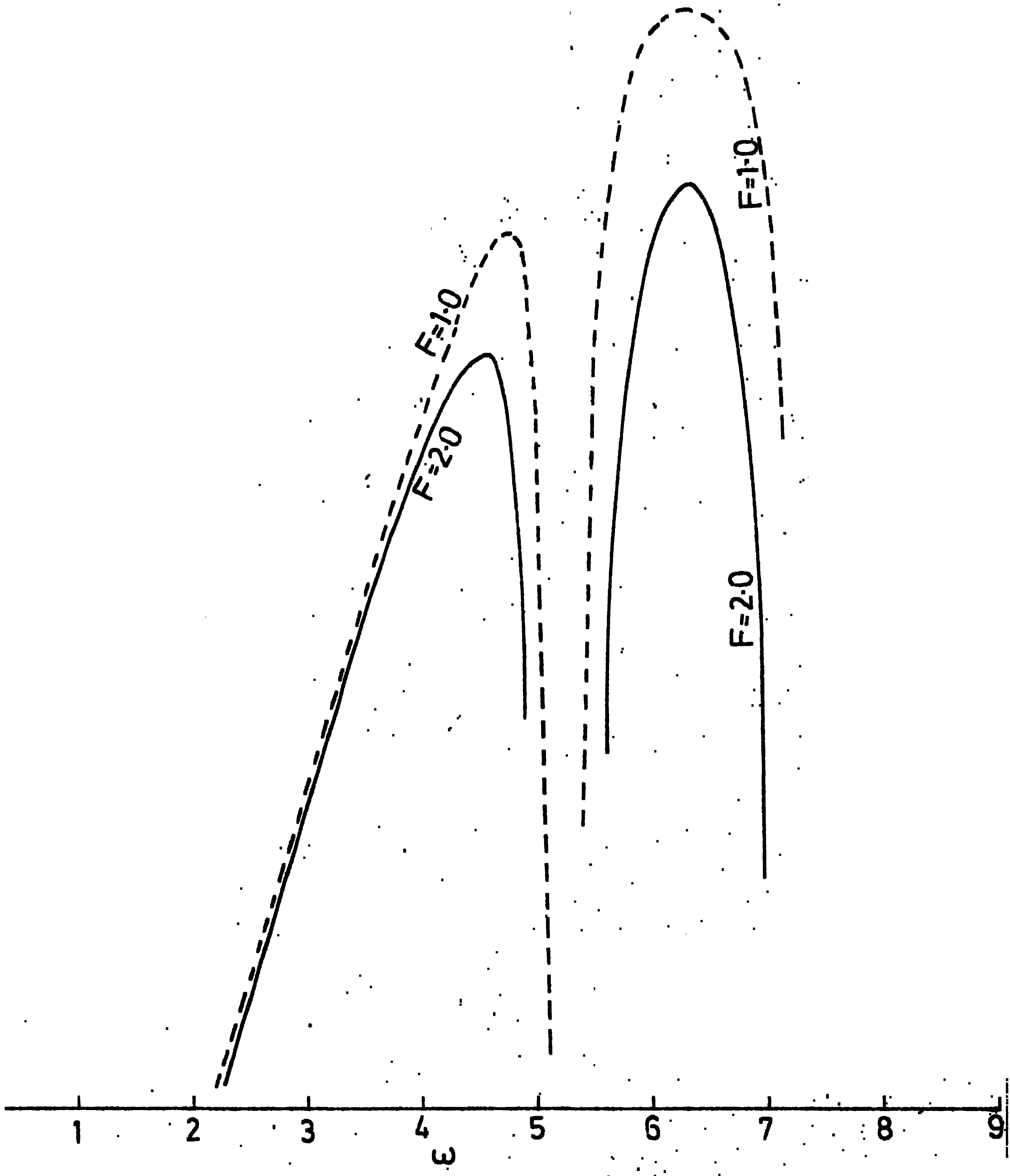
$$\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + \gamma = 0, \tag{3.34}$$

where

$$\mu\gamma = 2\cos\omega/5 + \alpha.$$

In a similar way the subharmonic of any order can be investigated.

Fig. 3.1 and 3.2 show the amplitude-frequency characteristic curves for different values of input amplitude F for subharmonic oscillation of order 2 and 5 respectively (without damping). The analysis reported in this chapter is valid only for small detuning (γ of order unity), that is the frequency of subharmonic oscillation of order q is nearly equal to the natural frequency of the system. Now $\alpha = -1.0$, corresponds to a natural frequency of the linear oscillatory system, $\omega_0 = \pi/3$. Thus the response curves in the Figs. 3.1 and 3.2 can be usefully used only for ω in the neighbourhood of $2\pi/3$ (Fig. 3.1) and $5\pi/3$ (Fig. 3.2). A more complete analysis for ω, quite different from $2\pi/3$ or $5\pi/3$ can be carried out using large detuning theory, which, however, is yet to be



FIG·3·1 AMPLITUDE VARIATION OF SUBHARMONIC OF ORDER 2

developed for discrete time systems.

Fig. 3.3 shows the variation of 3rd order subharmonic amplitude with the input frequency both with and without damping. In this case, for each value of frequency ω there are two values for the amplitude R_1. As in the case of continuous time systems it is perhaps possible to prove that one of the amplitudes is stable and the other is unstable. However, the stability analysis results in Hill type equations, whose analysis has not yet been developed for discrete time systems.

Next we will consider the possible superharmonic oscillations in this particular case.

(v) Superharmonic of order 2 :

Here $\omega_1 = 2\,\omega$.

Substituting this relation in (3.21) and collecting the coefficients of $\cos 2\omega\eta$ and $\sin 2\omega\eta$ terms, we obtain

$$a_1 = A(\tfrac{3}{4} R^2 + \tfrac{3}{2} P^2)$$

$$b_1 = B(\tfrac{3}{4} R^2 + \tfrac{3}{2} P^2).$$

With these values for a_1 and b_1 eqn. (3.18) takes the following form :

$$A(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + \gamma + N \cos \omega_1) - NB \sin \omega_1 = 0$$

$$B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + \gamma + N \cos \omega_1) + NA \sin \omega_1 = 0,$$

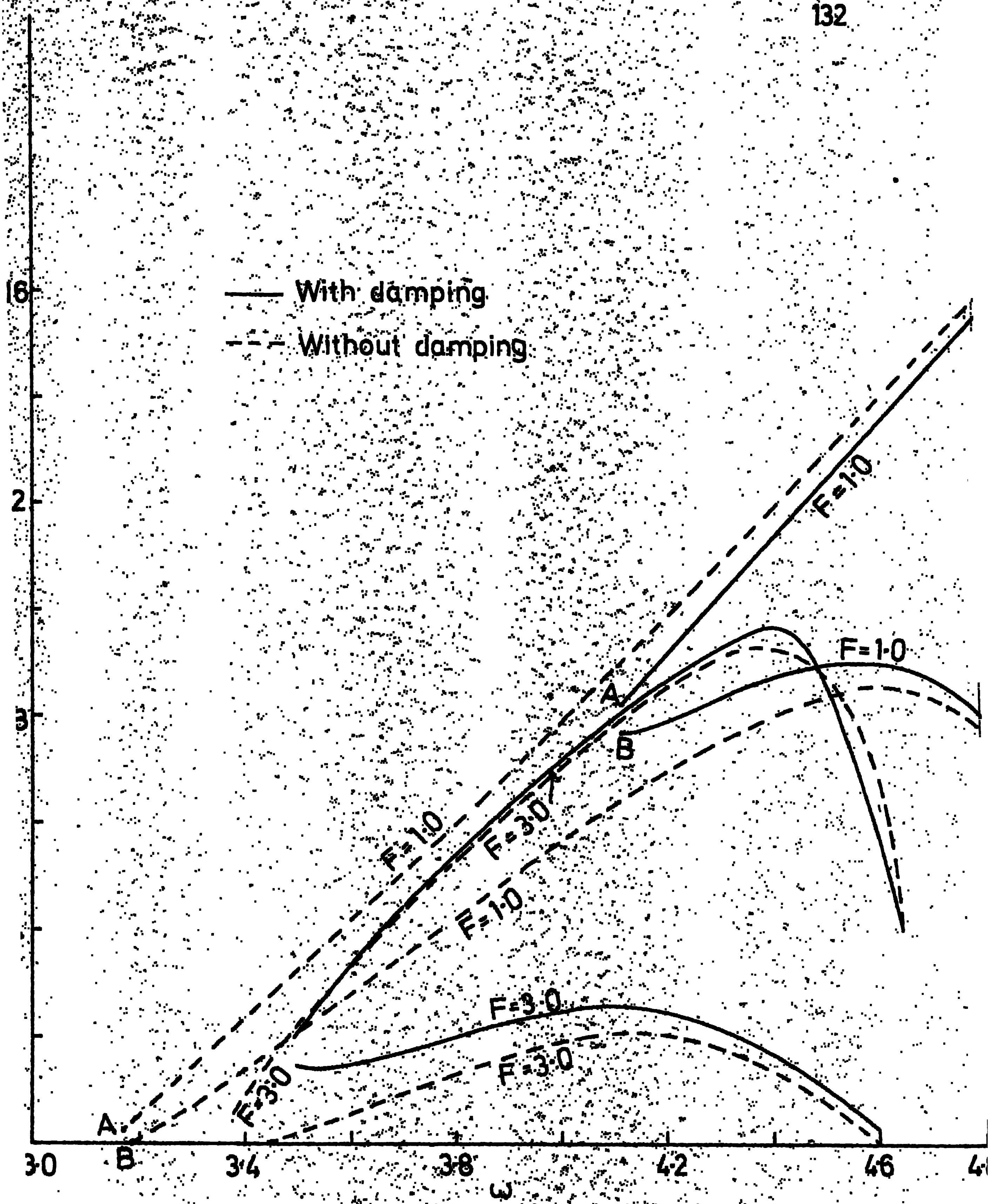


FIG.3·3 AMPLITUDE CHARACTERISTIC CURVES FOR SUBHARMONIC OF ORDER 3

from which, as before, there is no superharmonic oscillation of order 2 when the damping is nonzero. Whereas second order superharmonic solution is possible if $N = 0$, that is $\beta = 1.0$, Under this condition

$$\frac{3}{4}R^2 + \frac{3}{2}P^2 + \gamma = 0, \tag{3.35}$$

where $\mu\gamma = 2 \cos 2\,\omega + \alpha$.

The variation of amplitude with input frequency given by eqn. (3.25) is plotted in Fig. 3.4.

(vi) Superharmonic of order 3 :

In a similar manner as above

$$a_1 = A(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2) + \tfrac{1}{4}P^3$$
$$b_1 = B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2).$$

Then from eqn. (3.18)

$$A(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N \cos \omega_1 + \gamma) + \tfrac{1}{4}P^3 - NB \sin \omega_1 = 0 \tag{3.36}$$

$$B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N \cos \omega_1 + \gamma) + NA \sin \omega_1 = 0. \tag{3.37}$$

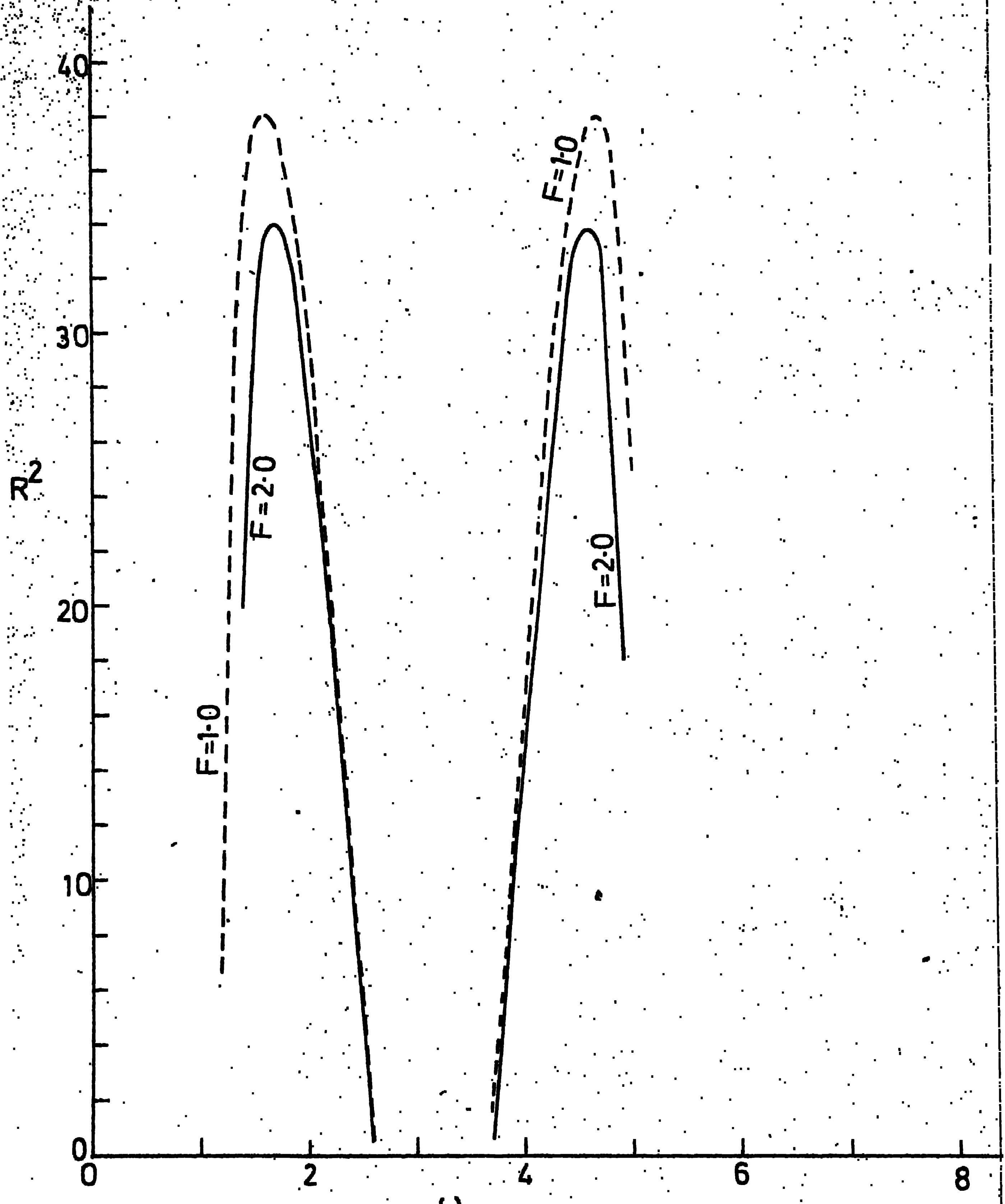
Further

$$AS - NB \sin \omega_1 = -\tfrac{1}{4} P^3$$

$$BS + NA \sin \omega_1 = 0,$$

where

$$S = \tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N \cos \omega_1 + \gamma,$$



FIG·3·4 AMPLITUDE CHARACTERISTIC FOR SUPERHARMONIC OF ORDER 2

Squaring and adding the above equations, we have

$$R^2(S^2 + N^2 \sin^2 \omega_1) = \frac{P^6}{16} .$$

Then substituting for S, we obtain

$$\frac{9}{16}R_1^3 + (\frac{9}{4}P^2 + \frac{3}{2} N \cos \omega_1 + \frac{3}{2}\gamma)R_1^2 + [\frac{9}{4}P^4 + N^2 + \gamma^2 + 3P^2\gamma + N \cos \omega_1(2\gamma + 3P^2)]R_1 - \frac{P^6}{16} = 0, \qquad (3.38)$$

where

$$R_1 = R^2$$

and

$$\omega_1 = 3\omega .$$

The variation of the amplitude of superharmonic of order 3 is given in the equation (3.38).

For dissipation free system, that is for $N = 0$, $(\beta = 1.0)$

$$\frac{9}{16}R_1^3 + (\frac{9}{4}P^2 + \frac{3}{2}\gamma) R_1^2 + (\frac{9}{4}P^4 + 3P^2\gamma + \gamma^2) R_1 - \frac{P^6}{16} = 0 \qquad (3.39)$$

The results obtained for various sub/super harmonic solutions using the discrete time perturbational technique are verified using 'Harmonic Balance' method in the following section.

3.5 Method of Harmonic Balance :

In continuous time system this is a method of wide

utility for obtaining periodic solution of a nonlinear differential equation. The study of super and subharmonic oscillations has also been carried out by various authors including Hayashi [4], Stoker [5] and Ku [7] adapting this technique. The very same technique is used in the present study to investigate the possible sub/super harmonic solutions in a strongly excited weakly nonlinear second order difference equations with polynomial nonlinearity. The model under study is same as that given in eqn. (3.1), that is

$$x(k+1) + \alpha x(k) + \beta x(k-1) + \mu f(x(k), x(k-1)) = F \cos \omega k \tag{3.40}$$

Now the procedure for obtaining the subharmonic solutions is given below :

Here the forcing frequency is assumed to be $p\omega$ instead of ω in the eqn. (3.40) for convenience. It will be noticed that, since the frequency of the external force is $p\omega$, the subharmonic components of order p has a frequency ω and it may be expressed as a linear combination of $\cos\omega k$ and $\sin\omega k$. Now the system eqn. (3.40) can be rewritten as :

$$x(k+1) + \alpha x(k) + \beta x(k-1) + \mu f(x(k), x(k-1)) = F \cos p\omega k. \tag{3.41}$$

The subharmonic solutions can be developed by assuming the solution x(k) to the eqn. (3.41) of the form

$$x(k) = A \cos\omega k + B \sin\omega k + P \cos p\omega k, \qquad (3.42)$$

where

A,B are amplitude functions of subharmonic oscillation of order p and P is the amplitude of the fundamental component. As in continuous time systems, the amplitude P may be approximated by

$$P = \frac{F}{[(\beta+1) \cos p\omega + \alpha]} \; .$$

Then from eqn. (3.42)

$$x(k+1) = (A \cos\omega + B \sin\omega) \cos\omega k - (A \sin\omega - B \cos\omega)$$
$$\sin\omega k + P \cos p\omega \,.\, \cos p\omega k - P \sin p\omega \,.\, \sin p\omega k.$$
$$x(k-1) = (A \cos\omega - B \sin\omega) \cos\omega k + (A \sin\omega + B \cos\omega)$$
$$\sin\omega k + P \cos p\omega \,.\, \cos p\omega k + P \sin p\omega \,.\, \sin p\omega k.$$

Substituting for x(k+1), x(k) and x(k-1) in the eqn. (3.41), we have

$$(A \cos\omega + B \sin\omega) \cos\omega k - (A \sin\omega - B \cos\omega) \sin\omega k$$
$$+ P \cos p\omega \,.\, \cos p\omega k - P \sin p\omega \,.\, \sin p\omega k$$
$$+ \alpha A \cos\omega k + \alpha B \sin\omega k + \alpha P \cos p\omega k$$
$$+ \beta(A \cos\omega - B \sin\omega) \cos\omega k + \beta(A \sin\omega + B \cos\omega)$$
$$\sin\omega k + \beta P \cos p\omega \,.\, \cos p\omega k + \beta P$$
$$\sin p\omega \,.\, \sin p\omega k + \mu a_p \cos\omega k + \mu b_p \sin\omega k$$
$$+ \mu a_2 \cos p\omega k = F \cos p\omega k,$$

where

a_p , b_p and a_2 are the coefficients of $\cos\omega\, k$, $\sin\omega\, k$ and $\cos p\omega\, k$ terms in Fourier series expansion of the nonlinear function in the eqn. (3.41).

Then equating the terms involving $\cos\omega\, k$, $\sin\omega\, k$ separately to zero, we obtain

$\cos\omega\, k$:

$$A\cos\omega + B\sin\omega + \alpha A + \beta(A\cos\omega - \beta\sin\omega) + \mu a_p = 0$$

$$A(\beta+1)\cos\omega + B(1-\beta)\sin\omega + \alpha A + \mu\, a_p = 0. \qquad (3.43)$$

$\sin\omega\, k$:

$$-A\sin\omega + B\cos\omega + \alpha B + \beta(A\sin\omega + B\cos\omega) + \mu b_p = 0$$

$$B(\beta+1)\cos\omega + A(\beta-1)\sin\omega + \alpha B + \mu\, b_p = 0. \qquad (3.44)$$

The eqns. (3.43) and (3.44) give the variation of subharmonic amplitude with the input frequency. The method of obtaining the amplitude of various orders of subharmonic oscillations are illustrated by considering a particular nonlinearity as indicated in the following example.

3.6 Example :

Consider the following nonlinear difference equation

$$x(k+1) + \alpha x(k) + \beta x(k-1) + \mu x^3(k) = F\cos p\omega\, k. \qquad (3.45)$$

Eqn. (3.45) is same as the eqn. (3.41) except for the nonlinear function, that is given by

$$f(\cdot) = x^3(k).$$

Then the interest is to find explicit expressions for a_p, b_p and a_2.

$$x^3(k) = (A \cos\omega k + B \sin\omega k + P \cos p\omega k)^3$$

$$= A^3\cos^3\omega k + 3 A^2B \cos^2\omega k. \sin\omega k$$

$$+ 3 AB^2 \cos\omega k. \sin^2\omega k$$

$$+ B^3 \sin^3 \omega k + 3P \cos p\omega k (A^2\cos^2\omega k$$

$$+ 2 AB \cos\omega k. \sin \omega k + B^2 \sin^2\omega k)$$

$$+ 3P^2 \cos^2 p\omega k (A \cos\omega k + B \sin\omega k) + P^3\cos^3 p\omega k.$$

Expanding and simplifying

$$x^3(k) = (\tfrac{3}{4} A^3 + \tfrac{3}{4}AB^2 + \tfrac{3}{2}AP^2) \cos \omega k + (\tfrac{3}{4} A^2B + \tfrac{3}{4}B^3 + \tfrac{3}{2} BP^2)$$

$$\sin \omega k + (\tfrac{1}{4} A^3 - \tfrac{3}{4} AB^2) \cos 3\omega k + (\tfrac{3}{4} A^2B - \tfrac{1}{4} B^3)$$

$$\sin 3\omega k + (\tfrac{3}{2} A^2P + \tfrac{3}{2} B^2P + \tfrac{3}{4}P^3) \cos p\omega k + \tfrac{1}{4}P^3 \cos 3 p\omega k$$

$$+ (\tfrac{3}{4} A^2P - \tfrac{3}{4}B^2P) \cos(p \omega + 2\omega)k + (\tfrac{3}{4}A^2P - \tfrac{3}{4}B^2P)$$

$$\cos(p\omega - 2\omega)k + \tfrac{3}{2}ABP \sin(2\omega + p\omega) k + \tfrac{3}{2}ABP$$

$$\sin(2\omega - p\omega)k + \tfrac{3}{4}AP^2 \cos(\omega + 2p\omega)k + \tfrac{3}{4}AP^2$$

$$\cos(\omega - 2p\omega)k + \tfrac{3}{4}BP^2 \sin(\omega + 2p \omega)k + \tfrac{3}{4}BP^2$$

$$\sin (\omega - 2p \omega)k. \qquad (3.46)$$

Now using the eqn. (3.46) the various orders of subharmonic oscillations are studied as follows.

(i) Subharmonic of order 2 :

Here $p = 2$.

Then from the eqn. (3.46)

$$a_2 = \frac{3}{4} A^3 + \frac{3}{4} AB^2 + \frac{3}{2} AP^2$$
$$= \frac{3}{4} AR^2 + \frac{3}{2} AP^2$$

$$b_2 = \frac{3}{4} A^2 B + \frac{3}{4} B^3 + \frac{3}{2} BP^2$$
$$= \frac{3}{4} BR^2 + \frac{3}{2} BP^2,$$

where

$$R^2 = A^2 + B^2.$$

Substituting a_2 and b_2 in eqns. (3.43) and (3.44), we obtain

$$A(\beta+1) \cos\omega - B(\beta-1) \sin\omega + \alpha A + \mu A(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2) = 0 \qquad (3.47)$$

$$B(\beta+1) \cos\omega + A(\beta-1) \sin\omega + \alpha\beta + \mu B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2) = 0, \qquad (3.48)$$

that is,

$$A[(\beta+1) \cos\omega + \alpha + \mu(\tfrac{3}{4} R^2 + \tfrac{3}{2}P^2)] - B(\beta-1) \sin\omega = 0$$

$$B[(\beta+1) \cos\omega + \alpha + \mu(\tfrac{3}{4} R^2 + \tfrac{3}{2}P^2)] + A(\beta-1) \sin\omega = 0.$$

Multiplying the first equation by B and the second by A and subtracting,

$$(\beta-1)\ R^2 \sin\omega\ = 0,$$

that is R = 0 for $(1-\beta) \neq 0$, which implies the amplitude of subharmonic oscillation is zero as long as the damping is present in the system. However with zero damping ($\beta = 1.0$) we have, $R \neq 0$ and subharmonic of order 2 can occur, that is

$$2 \cos\omega + \alpha + \mu\ (\tfrac{3}{4}R^2 + \tfrac{3}{2}\ P^2) = 0.$$

From which

$$\tfrac{3}{4}\ R^2 + \tfrac{3}{2}P^2 + \gamma = 0, \qquad (3.49)$$

where

$$P = \frac{F}{(2\ \cos\ 2\,\omega + \alpha)}$$

and

$$\mu\gamma = 2\ \cos\ \omega\ \ + \alpha.$$

It is interesting to note that the eqn. (3.49) and (3.24) are identical. In (3.24) the frequency $\omega_1 = \omega/2$ is the frequency of subharmonic oscillation of order 2, whereas in (3.49), the frequency ω is the frequency of subharmonic oscillation of order 2 (the input frequencies are ω and $2\,\omega$ respectively).

(ii) Subharmonic of order 3 :

Here $p = 3$.

Then as before

$$a_3 = \frac{3}{4} A^3 + \frac{3}{4} AB^2 + \frac{3}{2} AP^2 + \frac{3}{4} A^2 P - \frac{3}{4} B^2 P$$

$$= \frac{3}{4} AR^2 + \frac{3}{2} AP^2 + \frac{3}{4} P \ (A^2 - B^2)$$

$$b_3 = \frac{3}{4} BR^2 + \frac{3}{2} BP^2 - \frac{3}{2} ABP.$$

Combining these values with eqns. (3.43) and (3.44), we have

$$A(\beta+1) \cos \omega - B(\beta-1) \sin \omega + \alpha A + \mu A(\frac{3}{4}R^2 + \frac{3}{2}P^2)$$
$$+ \frac{3}{4} \mu P(A^2 - B^2) = 0 \qquad (3.50)$$

$$B(\beta+1) \cos\omega + A(\beta-1) \sin\omega + \alpha B + \mu B(\frac{3}{4}R^2 + \frac{3}{2}P^2)$$
$$- \frac{3}{2} \mu ABP = 0. \qquad (3.51)$$

The eqn. (3.50) can be rewritten as

$$A[\frac{(\beta+1) \cos \omega - 2 \cos \omega + 2 \cos \omega + \alpha}{\mu} + \frac{3}{4} R^2 + \frac{3}{2}P^2]$$

$$- \frac{B(\beta-1)}{\mu} \sin\omega + \frac{3}{4} P(A^2-B^2) = 0$$

$$A[\frac{(2 \cos \omega + \alpha)}{\mu} + \frac{(\beta-1)}{\mu} \cos \omega + \frac{3}{4} R^2 + \frac{3}{2}P^2]$$

$$- \frac{B(\beta-1)}{\mu} \sin \omega + \frac{3}{4} P(A^2 - B^2) = 0,$$

that is

$$A(\frac{3}{4}R^2 + \frac{3}{2}P^2 + N \cos\omega + \gamma) + \frac{3}{4} P(A^2-B^2) - BN \sin\omega = 0 \qquad (3.52)$$

and similarly from eqn. (3.51), we have

$$B(\tfrac{3}{4} R^2 + \tfrac{3}{2}P^2 + N \cos \omega + \gamma) - \tfrac{3}{2} ABP + AN \sin \omega = 0, \quad (3.53)$$

where

$$\mu\gamma = (2 \cos \omega + \alpha),$$

and $$N = \frac{\beta - 1.0}{\mu}.$$

The eqns. (3.52) and (3.53) are identical to eqns. (3.26) and (3.27) respectively. Following the same procedure as in eqns. (3.26) and (3.27) we have the following equation identical to the eqn. (3.29),

$$\frac{9}{16}R_1^2 + [\tfrac{27}{16} P^2 + \tfrac{3}{2} (N \cos \omega + \gamma)] R_1 + [\tfrac{9}{4} P^4 + 3\gamma P^2 + \gamma^2 + N(N + \cos \omega\, (2\gamma + 3P^2))] = 0.0$$

where

$$R_1 = R^2$$

and $$P = F/[(\beta+1) \cos 3\, \omega + \alpha].$$

(iii) Subharmonic of order 5 :

Proceeding as before with P = 5, we have

$$a_5 = \tfrac{3}{4} AR^2 + \tfrac{3}{2} AP^2$$

$$b_5 = \tfrac{3}{4} BR^2 + \tfrac{3}{2} BP^2,$$

that is with $\beta \neq 1.0$ (with damping), we have

$$A[(\beta+1)\cos\omega + \alpha + \mu(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2)] - B(\beta-1)\sin\omega = 0$$

$$B[(B+1)\cos\omega + \alpha + \mu(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2)] + A(\beta-1)\sin\omega = 0,$$

from which, as before for $\beta = 1.0$

$$\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + \gamma = 0. \qquad (3.54)$$

where

$$P = F/(2\cos 5\omega + \alpha).$$

Here again the eqn. (3.54) is identical to equation obtained by multiple scale perturbation technique.

As in the case of multiple time perturbation technique, the identification of super harmonic components by harmonic balance method is not straight forward, but the system equation is to be changed slightly. The analysis is carried out as follows.

Let the system equation be of the form

$$x(k+1) + \alpha x(k) + \beta x(k-1) + \mu f(x(k), x(k-1)) = F\cos\omega k, \quad (3.55)$$

where 'ω' is the input frequency. It is of interest to investigate superharmonic solution of order p ($p > 1$) and so the solution is assumed to be in the following form :

$$x(k) = A\cos p\omega k + B\sin p\omega k + P\cos\omega k, \qquad (3.56)$$

where

A and B are the amplitude of superharmonic solution of order P.

Then substituting the eqn. (3.56) in (3.55) and simplifying, we obtain

$$A(\beta+1) \cos p\omega - B(\beta-1) \sin p\omega + \alpha A + \mu\, a_p = 0 \qquad (3.57)$$

$$B(\beta+1) \cos p\omega \pm A(\beta-1) \sin p\omega + \alpha B + \mu\, b_p = 0 \,. \qquad (3.58)$$

As in the case of subharmonic solution, P, the amplitude of the fundamental component is approximated by

$$P = \frac{F}{(\beta+1)\cos\omega + \alpha} \,,$$

where

a_p and b_p are the coefficients of $\cos p\omega k$ and $\sin p\omega k$ terms in the Fourier series expansion of the nonlinear function $f(.,.)$ in (3.55).

Let $f(.\ ) = x^3(k)$. With this nonlinearity the expression given (3.46) can be used to determine the terms a_p and b_p simply by replacing $\omega = p\omega$ and $p\omega = \omega$. With this construction the following superharmonic solutions are studied.

(iv) Superharmonic of order 2 :

Superharmonic oscillation of order 2 is obtained by assuming p = 2 in the eqn. (3.46) after incorporating the above replacement. That is,

$$a_2 = \tfrac{3}{4}\, AR^2 + \tfrac{3}{2}\, AP^2$$

$$b_2 = \frac{3}{4} BR^2 + \frac{3}{2} BP^2.$$

Combining these values with eqns. (3.57) and (3.58)

$$A(\beta+1) \cos p\omega - B(\beta-1) \sin p\omega + \alpha A + \mu A(\frac{3}{4} R^2 + \frac{3}{2}P^2) = 0$$

$$B(\beta+1) \cos p\omega + A(\beta-1) \sin p\omega + \alpha B + \mu B(\frac{3}{4} R^2 + \frac{3}{2}P^2) = 0,$$

from which, as in the above cases, we have

$$\beta = 1.0$$

and $$\frac{3}{4} R^2 + \frac{3}{2} P^2 + \gamma = 0. \qquad (3.59)$$

where

$$\mu\gamma = 2 \cos 2\omega + \alpha$$

and

$$P = F/(2 \cos\omega + \alpha).$$

As expected the eqn. (3.59) is identical to the eqn. (3.35).

(v) Superharmonic of order 3 :

With p = 3, we have

$$a_3 = \frac{3}{4} AR^2 + \frac{3}{2} AP^2 + \frac{1}{4} P^3$$

$$b_3 = \frac{3}{4} BR^2 + \frac{3}{2} BP^2.$$

Combining these values with eqns. (3.57) and (3.58)

$$A(\beta+1) \cos p\omega - B(\beta-1) \sin p\omega + \alpha A + \mu A(\frac{3}{4}R^2 + \frac{3}{2}P^2) + \frac{1}{4} \mu P^3 = 0 \qquad (3.60)$$

$$B(\beta+1)\cos p\omega + A(\beta-1)\sin p\omega + \alpha B + \mu B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2) = 0, \quad (3.61)$$

adding and subtracting 2A cos $p\omega$ in the eqn. (3.60) and 2B cos $p\omega$ in the eqn. (3.61), we have

$$A(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N\cos p\omega + \gamma) + \tfrac{1}{4}P^3 - NB\sin p\omega = 0 \quad (3.62)$$

$$B(\tfrac{3}{4}R^2 + \tfrac{3}{2}P^2 + N\cos p\omega + \gamma) + NA\sin p\omega = 0. \quad (3.63)$$

The eqns.(3.62) and (3.63) can be compared with eqns. (3.36) and (3.37).

Then from eqns. (3.62) and (3.63), we obtain

$$\tfrac{9}{16}R_1^3 + (\tfrac{9}{4}P^2 + \tfrac{3}{2}N\cos p\omega + \tfrac{3}{2}\gamma)R_1^2 + [\tfrac{9}{4}P^4 + N^2 + \gamma^2 + 3P^2\gamma + N\cos p\omega\,(2\gamma + 3P^2)]R_1 - \frac{P^6}{16} = 0.0 \quad (3.64)$$

where

$$R_1 = R^2$$

and $\mu\gamma = 2\cos 3\omega + \alpha.$

In a similar way the other superharmonic oscillations can be investigated.

3.7 Discussion of Results :

The following observations can be made from the results obtained in the previous sections for cubic nonlinearity.

(i) An observation of the results obtained for various orders

of sub /superharmonic oscillations by (a) Discrete time perturbational technique and (b) Harmonic balance method, shows excellent agreement between the two methods.

(ii) The conditions deduced for the existence of sub/superharmonic are shown in Figs. 3.1 to 3.4. Figs. 3.1 and 3.2 show the variation of the amplitude of subharmonic solution of order 2 and 5 with input frequency. It must be pointed out that the existence of sub/superharmonic oscillations has been established only for the values of detuning parameter γ of $O(1)$, the above curves can not effectively be utilised for the following reason. As mentioned earlier, the natural frequency ω_0 of the linear system is $\pi/3$, and the value of γ is of order unity only when $\omega \approx 2\pi/3$ for order 2 and $\omega \approx 5\pi/3$ for order 5 subharmonic oscillations. The plots in Figs. 3.1 and 3.2 show that value of the detuning factor γ works out to be very high and hence with the present approach, small detuning theory cannot be used for detecting order 2 and order 5 subharmonic oscillations. The same argument is applicable for the superharmonic oscillation of order 2 with reference to the Fig. 3.4.

(iii) For subharmonic oscillation of order 3, the variation of amplitude with input frequency is shown in the Fig. 3.3 with zero and weak damping in the system. Here again the natural frequency ω_0 of the linear system is $\pi/3$. For this subharmonic oscillation the value of the detuning parameter γ

is expected to be of order unity only when the input frequency $\omega = \pi$. For $F = 1.0$ as well as for $F = 3.0$, with damping, large detuning theory is necessary for effective utilization of the response curves. However, with zero damping it is clear from the figure, small detuning theory analysis is possible and hence a subharmonic oscillation of order 3 can be sustained. As in the case of continuous time systems [4,117] the sub/superharmonic oscillations are highly dependent on initial conditions and furthermore since the equilibrium points in the $A(\tau) - B(\tau)$ space are expected to be centers/saddles (zero damping), hence the time response of the given system showing the subharmonic oscillation of order 3 is all the more difficult to obtain.

(iv) It is also to be pointed out that for the other values of F, namely F=2.0, 4.0, etc. (not shown in the Fig. 3.3), the structure of the curves are such that the impressed frequency, for admissible amplitude is further away from π than for $F = 3.0$.

(v) Another interesting observation is, that, for a given input amplitude ($F = 1.0$), the curve shown in the Fig. 3.3 is not closed as indicated by points A and B. It is expected that the two branches would meet, but this would entail finer increments in ω than what has been considered. In any case the closed curves would still be not in neighbourhood of π for effective use.

3.8 Conclusion :

The problem of sub/superharmonic oscillations in nonlinear discrete time systems has been studied in this chapter by means of discrete multiple scale perturbational technique as well as by harmonic balance method and conditions for existence of various orders of sub/superharmonic oscillations have been established. An example was presented with cubic nonlinearity and analysed in detail to demonstrate the use of the derived conditions. Proper initial conditions have, however, be selected to actually see the sub/superharmonic oscillations in the time response.

CHAPTER 4

PERIODIC OSCILLATIONS IN DIGITAL FILTERS

4.1 Introduction :

In the earlier chapters a class of second order nonlinear difference equations with and without external input were considered. Polynomial kind of nonlinearity was assumed and it was demonstrated that the multiple time perturbation technique can be applied to study the system behaviour. In this chapter, a second order difference equation with a nonlinearity other than the polynomial type is considered. As mentioned in the introductory chapter the description of operation of digital filters due to finite wordlength registers typically involves nonlinearities other than those of polynomial type. The various possible nonlinear effects that can occur in digital filters due to the finite wordlength available for the signal storage are briefly explained in the following section.

4.2 Nonlinear Phenomena in Digital Filters:

The wordlength of the signals in a digital system will in general increase after arithmetical operations (addition and multiplication). When storing signals after arithmetical operations the wordlength has to be reduced which is equivalent to nonlinear operations being performed on the signals. Two types of nonlinear phenomena are observed in digital filters :

(1) quantization and (2) overflow. Quantization is a word-length reduction in which the least significant bits are affected [79]. It will lead to an error that is proportional to the value corresponding with the least significant bit of the quantized signal. As such, the error will be quite small, but an error of this kind will be introduced during every sample period and the combined effect of all these errors may cause a severe distortion of the filter behaviour. Overflow occurs in digital filters, if after arithmatical operations, a word results that lies outside the range of the specified wordlength of the register. In this case the most significant bits of the signals must be affected before it can be stored in the register and this nonlinear phenomenon introduces large errors.

Both the quantization and overflow nonlinearities will sometimes generate sustained oscillations known as limit cycle oscillations in digital filters. Limit cycle oscillations due to quantization have relatively small amplitudes and they cannot be eliminated so easily. They have been studied in considerable detail using fixed point arithmetic [43, 79, 93, 111]. On the other hand the overflow nonlinear operation introduces a very large error and large amplitude limit cycle oscillations are then possible. Methods have been reported for analysis of these effects [76, 79, 84] and to suppress them to certain extent [96, 97]. As mentioned earlier, the

effects due to overflow nonlinearities are quite severe and their effects are analysed in detail in the present study. The influence of the quantization effects will assumed to be small and will not be considered in this analysis.

The study of overflow behaviour manifesting nonlinear oscillations because of adder overflow, and occuring in a second order section of a digital filter was begun in [76] and [80]. The system considered is shown in Fig. 4.1 and the various possible overflow nonlinearities are given in Fig. 4.2. It was shown in [76] that for a certain type of overflow nonlinearity; for example twos complement shown in Fig. 4.2(c), limit cycle oscillations can be sustained, for appropriately chosen initial conditions, under zero input situation. It was shown there that such oscillations can not exist in the zero input response of a second order digital filter employing saturation arithmetic depicted in Fig. 4.2(a). However, such limit cycle oscillations are possible with saturation nonlinearity if one takes the quantization effects also into account. Willson [84] among others has analysed a second order digital filter under zero input condition considering quantization effects. This analysis is most general and can be easily extended to wide varieties of overflow nonlinearities. Recently Mitra [92] made an observation that amongst digital filters employing the saturation arithmetic a fundamental difference exists between second order and higher order sections,

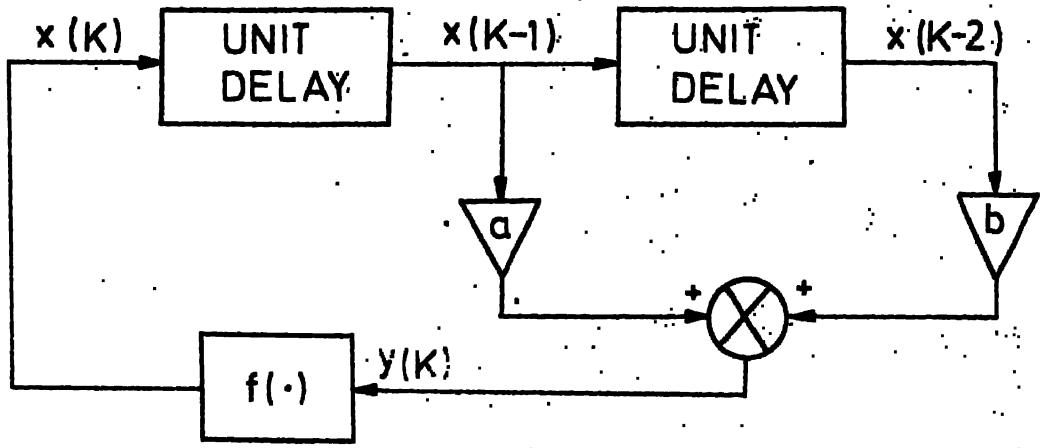
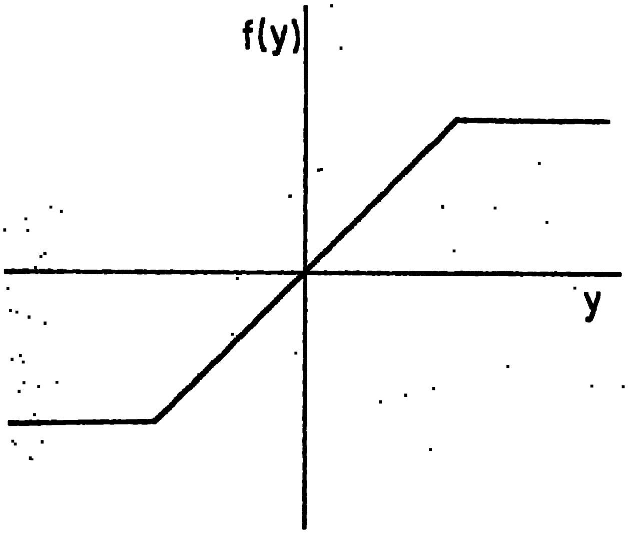
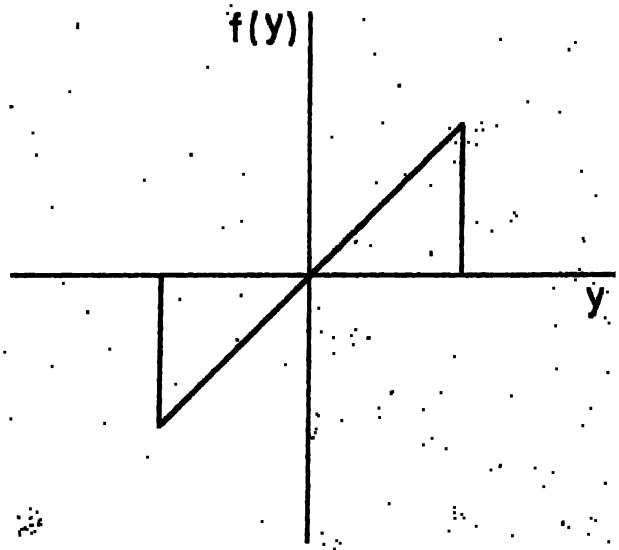


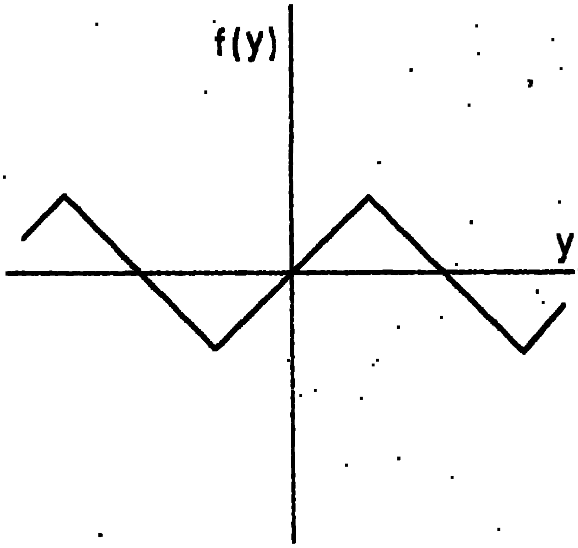
FIG.4.1 SECORD ORDER DIGITAL FILTER WITH ZERO INPUT



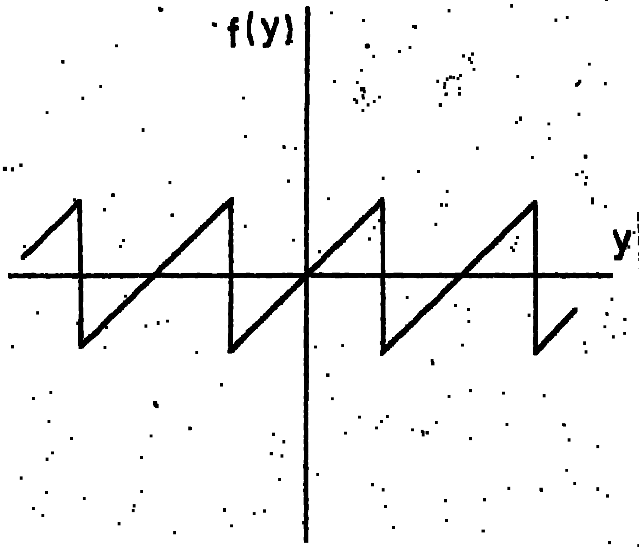
(a) Saturation arithmetic



(b) Zeroing arithmatic



(c) Generalised arithmetic



(d) Twos complement arithmati

namely the latter may sustain overflow oscillations. In all the above analysis it was assumed that the response of the linear model is asymptotically stable, that is the filter coefficients a and b have values determined by points that lie within the open triangular rigion shown in Fig. 4.3. An excellent review on effects of finite wordlength is given by Liu [40] and the limit cycle problem is given in [79, 98].

The available results on limit cycles concentrate on, conditions for the existence of limit cycles, estimate of limit cycle amplitude and frequency. A bulk of work has concentrated an second order digital filter with coefficient values inside the stability triangle. There has however been no work on limit cycle oscillations in second order digital filters for the coefficient values outside the stability triangle ABC given in Fig. 4.3. It is therefore of interest to study the limit cycle oscillations outside the stability triangle under force free situation. In this chapter, the study of overflow oscillations is extended for the parameter values outside the stability triangle with saturation overflow non-linearity.

The following are the main contributions in this study :

(i) The existence of limit cycles with different periods is demonstrated for parameter values outside the stability triangle. Regions delineating the parameter space

have been obtained where a particular kind of limit cycle oscillation with a known period is obtained.

(ii) To investigate a region in the parameter plane in which the output sequence decays monotonically for a set of values in the initial condition plane.

4.3 Limit Cycle Oscillation Outside the Stability Triangle :

It is evident that for parameter values outside the stability triangle, the linear response of the filter is unstable, which in turn results in nonlinear oscillations due to saturation nonlinearity. This nonlinear oscillation is however not always present for some values of [a,b] outside the stability triangle as demonstrated later in the analysis. In this section, the study is concentrated on identification of various regions outside the stability triangle, in the parameter plane where limit cycle oscillations of different periods are possible. The system under investigation is depicted in Fig. 4.4 and its mathematical description is given as

$$x(k+2) = f[ax(k+1) + bx(k)] \qquad (4.1)$$

where a and b are the filter coefficients or parameters and $f(\cdot)$ is the nonlinear function representing a saturation nonlinearity described by

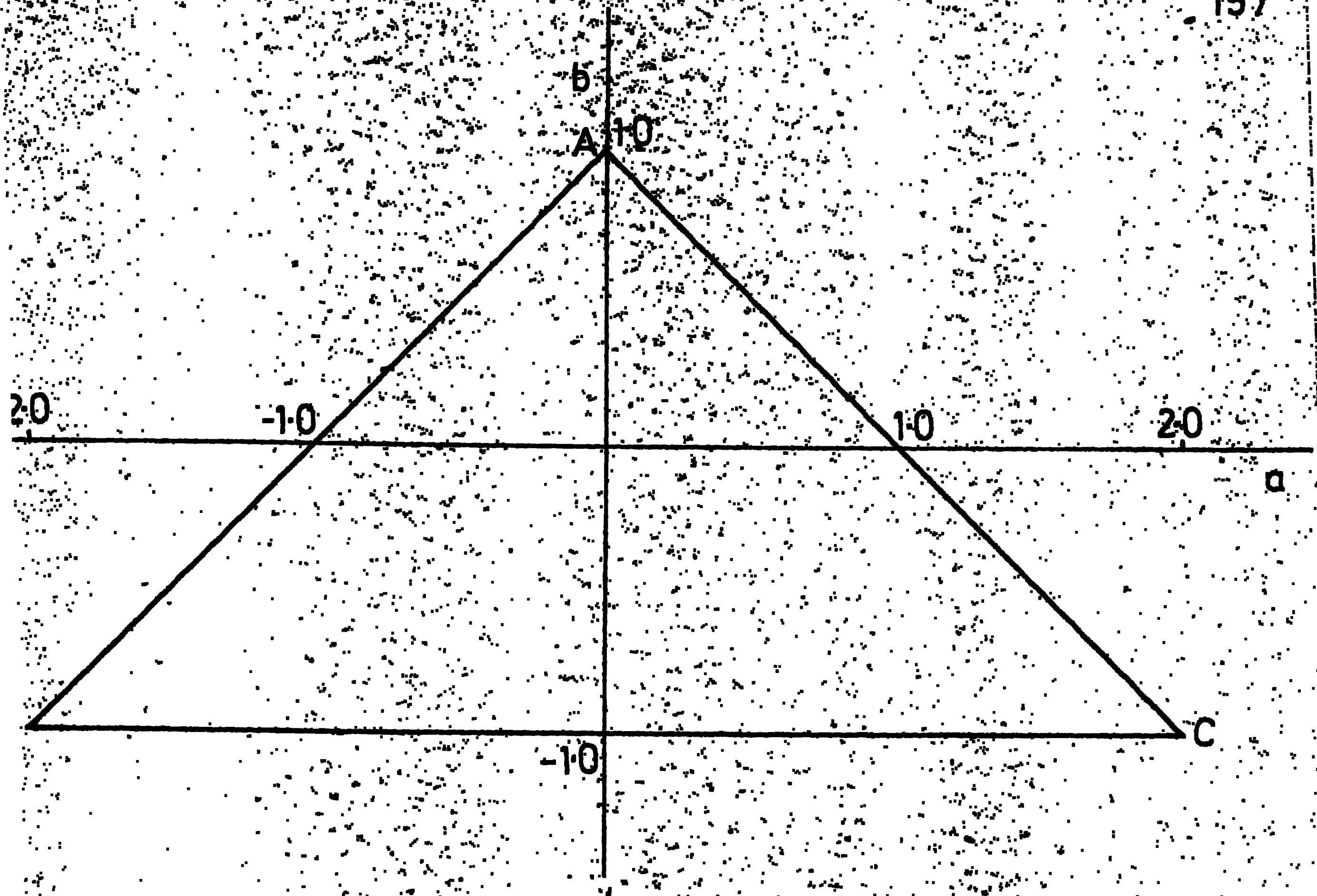


FIG.4·3 TRIANGULAR REGION IN a-b PLANE WITHIN WHICH THE LINEAR RESPONSE IS ASYMPTOTICALLY STABLE

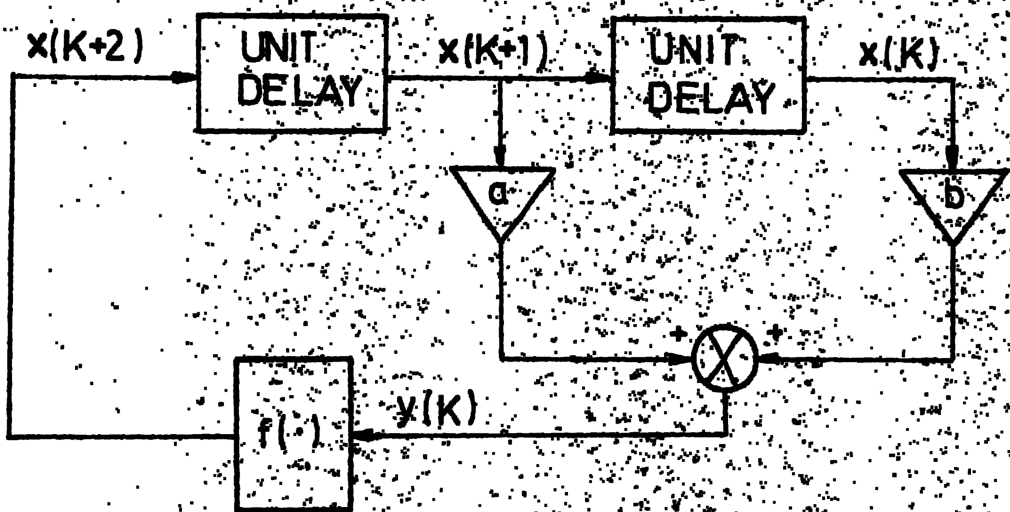


FIG.4·4 SYSTEM MODEL

$$f(\sigma) = \begin{cases} \sigma & |\sigma| \leq 1.0 \\ 1 & \sigma > 1.0 \\ -1 & \sigma < -1.0 \end{cases} \qquad (4.2)$$

with the schematic representation of this saturation nonlinearity shown in Fig. 4.2(a).

4.4 Analysis :

It is well known that within the stability triangle in the a-b parameter space the free system described by

$$x(k+2) = ax(k+1) + bx(k)$$

has an asymptotically decaying form for all initial conditions. Of interest is the nature of the solution for parameter locations outside the stability triangle and in particular the identification of regions where a limit cycle oscillation of given period L (denoted by P_L) is obtained. Three methods have been proposed for the above purpose, namely

(a) Employing actual saturation nonlinearity (Fig. 4.2a)

(b) Using an approximate polynomial for saturation nonlinearity and employing harmonic balance method.

(c) Using an approximate polynomial for saturation nonlinearity and applying discrete multiple scale perturbational technique [100].

4.4.1 Employing actual saturation nonlinearity :

The technique proposed by Ebert et al [76] for location of regions of P_1 and P_2 limit cycle oscillations with twos complement arithmetic (Fig. 4.2d) has been extended for the present study, that is for investigating regions outside the stability triangle in which P_L limit cycle oscillations are possible employing saturation overflow nonlinearity. Thus considering the second order digital filter given in eqn. (4.1) and the nonlinearity given in eqn. (4.2) the following P_L oscillations are discussed.

P_1 Limit cycle oscillations :

In this case the output of the filter x(k) is a constant [76]

Let $\quad x(k) = A$

Substituting in eqn. (4.1)

$$A = f[A(a + b)]$$

Then using the definition of the saturation nonlinearity described in eqn. (4.2).

If $[A(a+b)] > 1.0$

$$A = 1.0 \tag{4.3}$$

If

$$[A(a+b)] < -1.0$$

$$A = -1.0 \tag{4.4}$$

and if $|A(a+b)| \leq 1.0$

$$A = A\,(a+b)$$

i.e. $a+b = 1.0$ (4.5)

the above expression in (4.5) represents the linear operation of the filter.

From eqns. (4.3) and (4.4), we obtain

$$a + b > 1.0 \tag{4.6}$$

The region in the a-b plane where a constant output of magnitude unity exists is given by the inequality (4.6).

P_2 limit cycle oscillation :

In general the period two periodic oscillation is expressed as

$$x(k) = A + B \cos \pi k \tag{4.7}$$

where,

with $A \neq 0$ and $B \neq 0$, the oscillation is about a constant d-c value, whereas with $A = 0$ and $B \neq 0$ the oscillation has no d-c bias. Then from eqn. (4.7)

$$x(0) = A + B$$

$$x(1) = A - B$$

and this sequence $(A + B, A - B)$ repeats for all even and odd values of k respectively.

Substitution of the values $x(0)$ and $x(1)$ in eqn. (4.1) leads

$$A + B = f[a(A-B) + b(A+B)]$$
$$A - B = f[a(A+B) + b(A-B)]$$

from the above

if $[a(A-B) + b(A+B)] > 1.0$

$$A + B = 1.0 \tag{4.8}$$

if $[a(A+B) + b(A-B)] > 1.0$

$$A - B = 1.0 \tag{4.9}$$

if $[a(A+B) + b(A-B)] < -1.0$

$$A - B = -1.0 \tag{4.10}$$

if $[a(A-B) + b(A+B)] < -1.0$

$$A + B = -1.0 \tag{4.11}$$

whereas

if $|a(A+B) + b(A-B)| \leq 1.0$

$$A-B = a(A+B) + b(A-B) \tag{4.12}$$

and

if $|a(A-B) + b(A+B)| \leq 1.0$

$$A + B = a(A-B) + b(A+B) \tag{4.13}$$

Finally

if $[a(A+B) + b(A-B)] = 0$

$$A - B = 0 \tag{4.14}$$

if $[a(A-B) + b(A+B)] = 0$

$$A + B = 0 \ . \tag{4.15}$$

Now the required region in the a-b plane can be obtained by considering the above equations as follows :

Solving eqns. (4.8) and (4.9) we get,

$$A = 1.0$$

$$B = 0.0$$

$$\text{with} \quad a+b > 1.0$$

and similarly considering eqns. (4.10) and (4.11)

$$A = -1.0$$

$$B = 0.0$$

$$\text{with } a+b = > 1.0$$

which in fact shows that P_1 limit cycle oscillation of unit magnitude is possible in the region $a+b > 1.0$ which is the same result obtained in eqn. (4.6),

Further considering eqns. (4.8) and (4.10), the following relations are obtained

$$A = 0.0$$

$$B = 1.0$$

$$\text{and } b-a > 1.0$$

and in a similar way with equations (4.9) and (4.11) we obtain,

$$A = 0.0$$

$$B = -1.0$$

$$\text{and} \quad b-a > 1.0$$

which implies that limit cycle oscillations of period two without d.c. component is possible in the region

$$b - a > 1.0 . \tag{4.16}$$

Further by combining eqns. (4.8) and (4.14) we get

$$A = 0.5$$
$$B = 0.5$$
$$a = 0.0$$
and $$b > 1.0$$

Similarly the following conditions are obtained by considering eqns. (4.9) and (4.15)

$$A = 0.5$$
$$B = -0.5$$
$$a = 0.0$$
$$b > 1.0$$

Also from eqns. (4.10) and (4.16)

$$A = -0.5$$
$$B = 0.5$$
$$a = 0.0$$
$$b > 1.0$$

further from eqns. (4.11) and (4.14)

$$A = -0.5$$
$$B = -0.5$$
$$a = 0.0$$
$$b > 1.0$$

which shows limit cycle oscillation of period two with d.c. bias, of the form $x(k) = \pm 0.5 \pm 0.5 \cos\pi k$ is possible only on the line $b > 1.0$ and $a = 0.0$.

P_3 limit cycle oscillation :

In this case the general solution is assumed to be of the form

$$x(k) = A + B \cos \frac{2\pi k}{3} + c \sin \frac{2\pi k}{3}$$

then

$$x(o) = A + B$$
$$x(1) = A - B/2 + \frac{\sqrt{3}}{2} c$$
$$x(2) = A - B/2 - \frac{\sqrt{3}}{2} c$$

and this sequence repeats for all $k \geq 3$.

Then the substitution of the above relations in eqn. (4.1) leads the following equations.

$$x(2) = A - \frac{B}{2} - \frac{\sqrt{3}}{2}c = f[a(A - \frac{B}{2} + \frac{\sqrt{3}}{2} c) + b(A + B)]$$

$$x(3) = A + B = f[a(A - B/2 - \frac{\sqrt{3}}{2}c) + b(A - B/2 + \frac{\sqrt{3}}{2}c)]$$

$$x(4) = A - \frac{B}{2} + \frac{\sqrt{3}}{2}c = f[a(A+B) + b(A - B/2 - \frac{\sqrt{3}}{2}c)]$$

if

$$a(A - \frac{B}{2} + \frac{\sqrt{3}}{2} c) + b(A + B) > 1.0$$

$$A - \frac{B}{2} - \frac{\sqrt{3}}{2} c = 1.0 \qquad (4.17)$$

if $a(A - \frac{B}{2} - \frac{\sqrt{3}}{2} c) + b(A - \frac{B}{2} + \frac{\sqrt{3}}{2} c) > 1.0$

$$A + B = 1.0 \tag{4.18}$$

if $a(A+B) + b(A - \frac{B}{2} - \frac{\sqrt{3}}{2} c) > 1.0$

$$A - \frac{B}{2} + \frac{\sqrt{3}}{2} c = 1.0 \ . \tag{4.19}$$

Solving (4.17), (4.18) and (4.19) we get

$$A = 1.0$$

$$B = 0.0$$

$$C = 0.0$$

that is the solution $x(k) = 1.0$ and the corresponding regin is

$$a + b > 1.0$$

which is same as (4.6).

Let

$$a(A - \frac{B}{2} + \frac{\sqrt{3}}{2} c) + b(A + B) = 0.0$$

$$A - \frac{B}{2} - \frac{\sqrt{3}}{2} c = 0.0 \tag{4.20}$$

and

$$a(A+B) + b(A - \frac{B}{2} - \frac{\sqrt{3}}{2} c) < -1.0$$

$$A - \frac{B}{2} + \frac{\sqrt{3}}{2} c = -1.0 \ . \tag{4.21}$$

Solving eqns. (4.18), (4.20) and (4.21) we obtain

$$A = 0.0$$

$$B = 1.0$$

$$C = -\frac{1}{\sqrt{3}}$$

and the corresponding regin in the a – b plane is

$$a = b$$
$$b < -1.0$$
$$a < -1.0\,. \qquad (4.22)$$

Eqn. (4.22) gives the straight line on which limit cycle oscillation of period 3 is possible without d.c. component.

Further for

$$\left| a\left(A - \frac{B}{2} + \frac{\sqrt{3}}{2}c\right) + b(A + B)\right| \leq 1.0$$

$$A - \frac{B}{2} - \frac{\sqrt{3}}{2}c = a\left(A - \frac{B}{2} + \frac{\sqrt{3}}{2}c\right) + b(A + B) \qquad (4.23)$$

then the conditions given in (4.18), (4.21) and (4.23) lead to the following region where a limit cycle oscillation of period 3 with d.c. component can be sustained.

$$|b - a| < 1.0$$
$$a(b-a) - b > 1.0$$
$$a + b(b-a) < -1.0 \qquad (4.24)$$

P_4 limit cycle oscillation :

The output sequence takes the form

$$x(k) = A + B\cos\frac{\pi k}{2} + C\sin\frac{\pi k}{2}$$

then

$$x(o) = A + B$$
$$x(1) = A + C$$
$$x(2) = A - B$$
$$x(3) = A - C$$

and this sequence repeats for all $k \geq 4$.

Then,

$$x(2) = A - B = f[a(A+C) + b(A+B)]$$
$$x(3) = A - C = f[a(A-B) + b(A+C)]$$
$$x(4) = A + B = f[a(A-C) + b(A-B)]$$
$$x(5) = A + C = f[a(A+B) + b(A-C)]$$

These equations combined with the properties of the saturation nonlinearity, the region in which a limit cycle oscillation of period 4 is possible is given by

$$\begin{aligned} a + b &< -1.0 \\ a - b &> -1.0 \end{aligned} \qquad (4.25)$$

In a similar way a region in the a - b plane in which, a period 6 limit cycle oscillation can be sustained is given by

$$\begin{aligned} |a + b| &< 1.0 \\ a - b(a+b) &> 1.0 \\ a(a+b) + b &< -1.0 \end{aligned} \qquad (4.26)$$

It is interesting to observe that the region of P_6 oscillation is identical to region of P_3 oscillation mirrored around a=0.0.

The investigation of regions corresponding to limit cycle oscillation of period 5 and other periods are possible but due to their fine structure in the a-b plane (checked by extensive simulation) such regions are not investigated in detail. The various regions showing different periods of limit cycle oscillations described the equations (4.6), (4.16), (4.24), (4.25) and (4.26) are sketched in Fig. 4.6. I in the Fig. 4.6 indicates a region where for small charges in $[a,b]$ the response period charges drastically. The above regions can also be obtained by considering a polynomial approximation to the saturation nonlinearity. This polynomial approximation is a valid approximation only for certain range of the variable. This approach is discussed in the following section.

4.4.2 Polynomial approximation for the saturation nonlinearity

This method considers a polynomial approximation to the saturation nonlinearity. The coefficients of the polynomial being a function of the range of interest D of the dependent variable are obtained by a least squares fit. Thus,

$$f(y) = \sum_{i=1}^{N} a_i\, y^{2i-1}, \quad \text{with} \quad |y| \leq D.$$

Then for N=4, the coefficients are $a_1 = 1.16253$, $a_2 = -0.2850$, $a_3 = 0.03643$, $a_4 = -0.00167$ for $D = 3.0$ (see Appendix C).

that is

$$f(y) = 1.16253y - 0.285y^3 + 0.03643y^5 - 0.00167y^7, \text{ with}$$

$$|y| \leq 3.0 . \qquad (4.27)$$

The saturation nonlinearity and its polynomial approximation (dotted lines) are indicated in Fig. 4.5.

The approximate polynomial expression given in (4.27) can be used to investigate the regions in the a-b plane in which various P_L limit cycle oscillations are possible. The analysis proceeds as follows.

P_1 limit cycle oscillation :

Let $x(k) = A$, a constant

From the system configuration shown in Fig. 4.4.

$$y(k) = ax(k+1) + bx(k)$$

$$y = A(a+b) .$$

Again from Fig. 4.4,

$$x(k+2) = f[y(k)]$$

that is $A = f(y) = 1.16253y - 0.285y^3 + 0.03643y^5 - 0.00167y^7.$

Substituting for y

$$A = 1.16253A(a+b) - 0.285\, A^3(a+b)^3 + 0.03643A^5(a+b)^5 - 0.00167A^7(a+b)^7$$

with

$$|A(a+b)| \leq 3.0$$

the above equation can be rewritten as

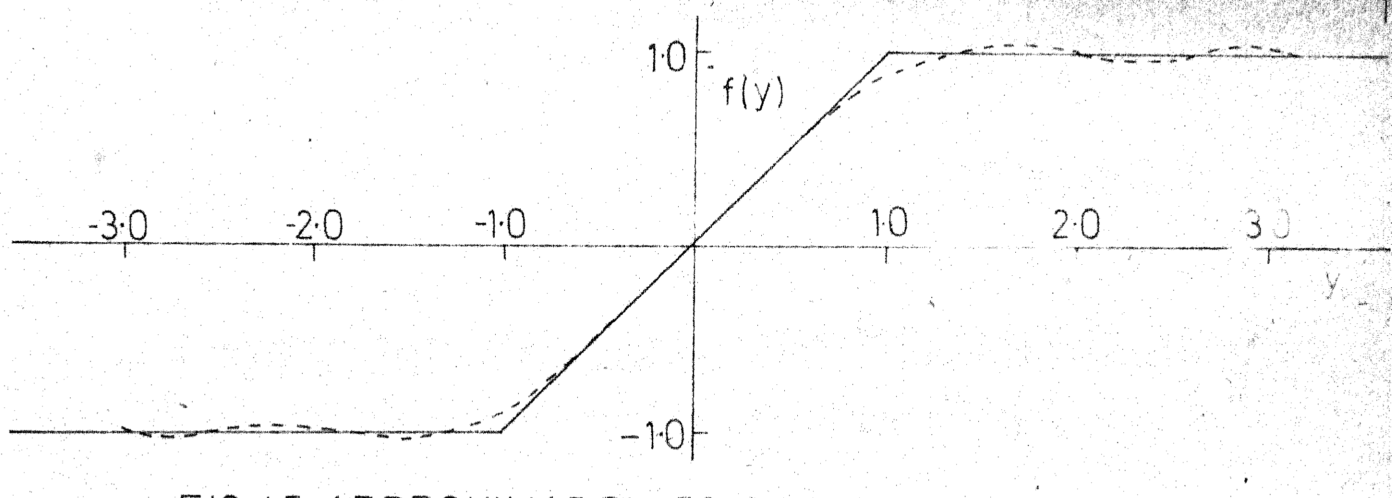


FIG. 4.5 APPROXIMATION TO SATURATION NONLINEARITY

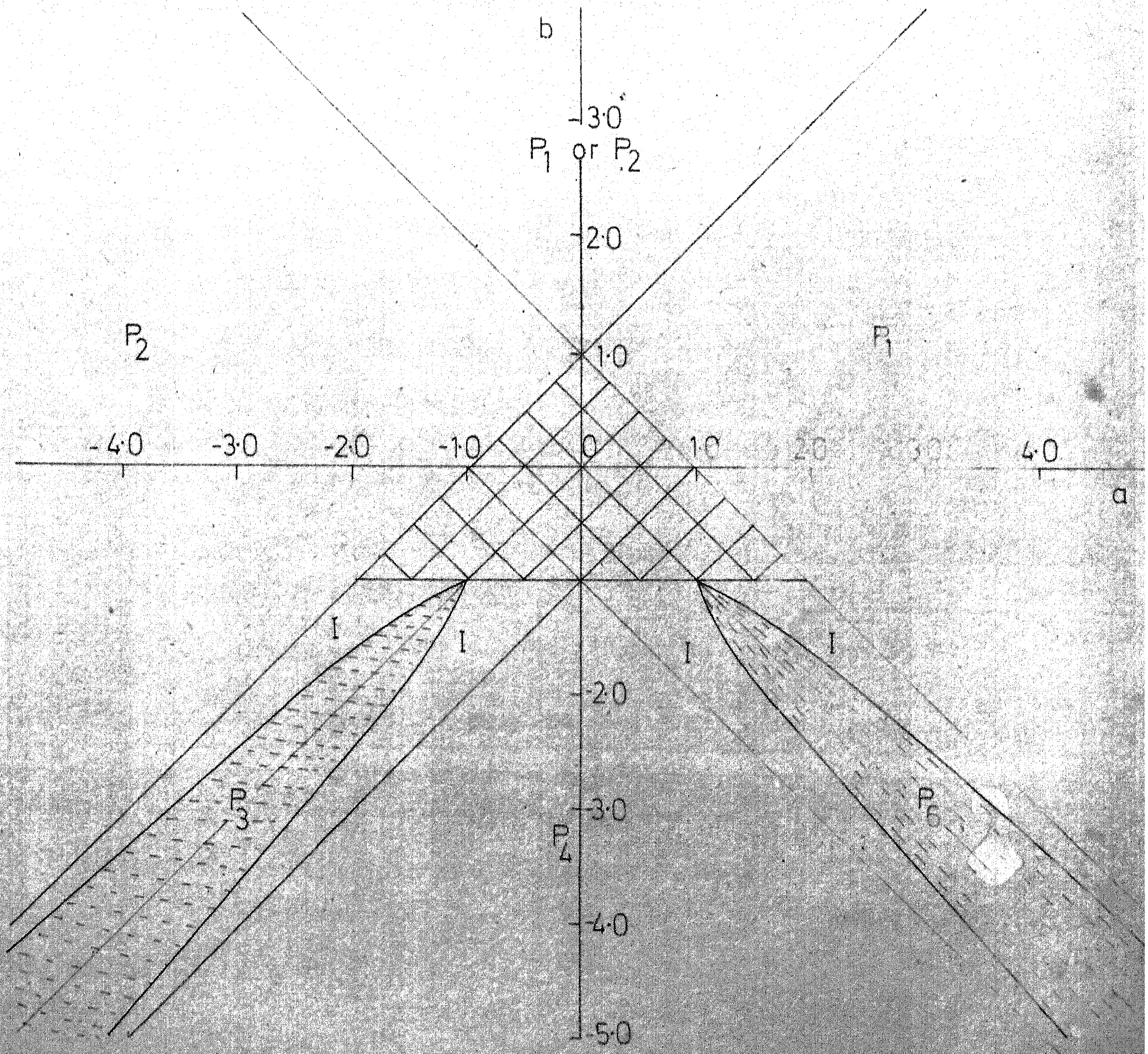


FIG. 4.6 REGIONS SHOWING P_L LIMIT CYCLE OSCILLATIONS

$$1.16253 - 0.285\, A^2(a+b)^2 + 0.03643A^4(a+b)^4 - 0.00167A^6(a+b)^6$$

$$= \frac{1}{a+b} \; . \tag{4.28}$$

Substituting the maximum bound on $|A(a+b)|$ which is 3.0,

$$a + b = 3.02 \; . \tag{4.29}$$

This gives the maximum bound on (a+b).

In a similar way the minimum bound on (a+b) can be obtained by substituting $|A(a+b)| = 1.0$,

$$a + b = 0.98 \; . \tag{4.30}$$

From equations (4.29) and (4.30) the region in which a limit cycle oscillation of period one is possible is given by

$$0.98 < a + b < 3.02 \; .$$

It is also noticed that as D becomes larger the upper bound on a+b also becomes greater and the complete region in the a-b plane for D $\to\infty$ is given by

$$a + b > 0.98 \tag{4.31}$$

P_2 limit cycle oscillation :

In this case the output sequence assumes the form

$$\begin{aligned} x(k) &= A + B \cos \pi k \\ y(k) &= ax(k+1) + bx(k) \\ &= A(a+b) + (b-a)\, B \cos \pi k \\ &= P + Q \cos \pi k \end{aligned}$$

where

$$P = A(a+b)$$

$$Q = B(b-a)$$

Considering the approximate polynomial for saturation nonlinearity

$$x(k+2) = f(y) = 1.16253y - 0.285y^3 + 0.03643y^5 - 0.00167y^7$$

$$\text{with } |y| = |P + Q\cos\pi k| \lesssim 3.0$$

that is

$$A + B\cos\pi k = 1.16253\,(P+Q\cos\pi k) - 0.285\,(P + Q\cos\pi k)^3$$
$$+ 0.03643\,(P + Q\cos\pi k)^5 - 0.00167\,(P + Q\cos\pi k)^7.$$

Expanding and equating the constant term and the coefficient of $\cos\pi k$ term separately using the following trigonometric relations

$$\cos^n \pi k = \cos\pi k \qquad \text{for } n \text{ odd}$$
$$= 1.0 \qquad \text{for } n \text{ even}$$

we obtain

$$P[1.16253 - 0.285\,(P^2 + 3Q^2) + 0.03643\,(P^4 + 10\,P^2Q^2 + 5\,Q^4)$$
$$- 0.00167\,(P^6 + 21\,P^4Q^2 + 35\,P^2Q^4 + 7\,Q^6] = A$$

$$Q[1.16253 - 0.285\,(3P^2 + Q^2) + 0.03643\,(5P^4 + 10P^2Q^2 + Q^4)$$
$$- 0.00167\,(7P^6 + 35P^4Q^2 + 21P^2Q^4 + Q^6] = B. \qquad (4.32)$$

Case i

If $A = 0$, then $P = 0$ and for $a \neq b$

and also $|y| \leq 3.0$ implies $|Q| \leq 3.0$.

Then eqn. (3.32) reduces to

$$1.16253 - 0.265\, Q^2 + 0.03643\, Q^4 - 0.00167\, Q^6 = \frac{1}{b-a} .$$

Substituting the bounds on Q

$$0.98 < b - a < 3.02 .$$

Then as indicated earlier the region in a-b plane (for $D \to \infty$) in which P_2 limit cycle oscillations are possible is

$$b - a > 0.98 . \qquad (4.33)$$

Case ii

If $A \neq 0$ and $B = 0$

Conditions are same as for P_1 limit cycle oscillation.

Case iii

If $A \neq 0$ and $B \neq 0$

Then the equations in (4.32) can be rewritten as

$$1.16253 - 0.285\,(P^2+3Q^2) + 0.03643\,(P^4 + 10P^2Q^2 + 5Q^4) - 0.00167\,(P^6 + 21P^4Q^2 + 35P^2Q^4 + 7Q^6) = \frac{1}{a+b} \qquad (4.34)$$

$$1.16253 - 0.285\,(3P^2+Q^2) + 0.03643\,(5P^4+10P^2Q^2+Q^4) - 0.00167\,(7P^6+35P^4Q^2 + 21P^2Q^4 + Q^6) = \frac{1}{b-a} . \qquad (4.35)$$

Solution to eqns. (4.34) and (4.35) is possible only if $a = 0$. Under this condition $P = Q$, which implies $A = B$, that is

$$x(k) = A(1 + \cos \pi k)$$

which is of the same form obtained by the first method. In a similar way P_3, P_4 and P_6 limit cycle oscillations can be obtained.

The method of obtaining the above regions using the discrete multiple time perturbation technique employed in Chapter 2 is now given.

4.4.3 Approximate polynomial and multiple scale perturbation method :

A multiple time perturbation technique has been proposed in [100] to study a class of discrete time system with polynomial kind of nonlinearity for obtaining an approximate closed form solution. The very same technique is employed here to classify different regions in a-b parameter plane in which different P_L oscillations are possible by suitably rearranging the system equation with polynomial approximation for the nonlinearity.

The pertinent filter equation to be analysed is given by

$$x(k+2) = 1.16253y - 0.285y^3 + 0.03643y^5 - 0.00167y^7$$

where $y = y(k) = a\,x(k+1) + bx(k)$.

The above equation can conveninntly be written as

$$x(k+2) + a_1 x(k+1) + a_2 x(k) = \mu g[x(k), x(k+1)] \qquad (4.36)$$

where g is a known function of indicated variable and is given by

$$\mu g(\cdot) = (1.16253a + a_1)\, x(k+1) + (1.16253b + a_2)\, x(k) - 0.285 [ax(k+1) + bx(k)]^3 + 0.03643[ax(k+1)+bx(k)]^5 - 0.00167[ax(k+1) + bx(k)]^7 \qquad (4.37)$$

and a_1 and a_2 are constants to be selected such that the linear response for $\mu = 0$ is periodic with known period.

P_1 limit cycle oscillation

From the properties of the stability triangle, for $a_1 = -\alpha$ and $a_2 = -\beta$ such that $\alpha + \beta = 1.0$, the response of the linear model is given by

$$x(k) = A + B(-\beta)^k \qquad (4.38)$$

With the above values for a_1 and a_2 eqn. (4.36) can be rewritten as

$$x(k+2) - \alpha x(k+1) - \beta x(k) = \mu g(x(k), x(k+1) \qquad (4.39)$$

where

$$\mu g(\cdot) = (1.16253a - \alpha)\, x(k+1) + (1.16253b - \beta)\, x(k) - 0.285[ax(k+1) + bx(k)]^3 + 0.03643\, [ax(k+1) + bx(k)]^5 - 0.00167\, [ax(k+1) + b\, x(k)]^7. \qquad (4.40)$$

The eqn. (4.39) is a weakly nonlinear difference equation for which multiple time perturbation technique can be applied. Here the dependent variable x is a function of two independent variables namely η and τ and so the variable x and its differences can be expressed in terms of η and τ as follows.

$$x(k) = x(\eta,\tau)$$
$$x(k+1) = x(\eta+1,\tau) + \mu x(\eta,\tau+1) - \mu\, x(\eta,\tau)$$
$$x(k+1) = x(\eta+2,\tau) + 2\mu\, x(\eta+1,\ \tau+1) - 2\mu x(\eta+1,\tau). \qquad (4.41)$$

It is to be noted that eqn. (4.41) can easily be obtained by substituting $T = \Delta$, the forward difference operator and $S_n = \omega_n = 0$ in eqns. (2.24) and (2.25). In this present study the analysis is carried out through two time scaling model obtained using the forward difference operator, since the interest is to obtain conditions under steady state situation.

The solution x(k) is also expressed in the following asymptotic series as :

$$x(k) = x(\eta,\tau) = \sum_{j=0} \mu^j x_j(\eta,\tau) . \qquad (4.42)$$

Substituting eqns. (4.41) and (4.42) in (4.39) and collecting the coefficients of μ^1, μ^1, μ^2 etc. terms and equating them separately to zero, we obtained the following set of linear difference equations :

$$x_o(\eta+2,\tau) - \alpha\, x_o(\eta+1,\tau) - \beta\, x_o(\eta,\tau) = 0 \tag{4.43}$$

$$\begin{aligned} x_1(\eta+2,\tau) - \alpha\, x_1(\eta+1,\tau) - \beta\, x_1(\eta,\tau) &= \alpha[x_o(\eta,\tau+1) - x_o(\eta,\tau)] \\ &+ \frac{(1.16253\, a - \alpha)}{\mu}\, x_o(\eta+1,\tau) + \frac{1.16253b-\beta}{\mu}\, x_o(\eta,\tau) - \frac{0.285}{\mu} \\ &[a\, x_o(\eta+1,\tau) + b\, x_o(\eta,\tau)]^3 \\ &+ \frac{0.03643}{\mu}[a\, x_o(\eta+1,\tau) + b x_o(\eta,\tau)]^5 \\ &- \frac{0.00167}{\mu}[a x_o(\eta+1,\tau) + b x_o(\eta,\tau)]^7 . \end{aligned} \tag{4.44}$$

Solution to eqn. (4.43) is known as generating solution and it is of the form

$$x_o(\eta,\tau) = A(\tau) + B(\tau) * (-\beta)^\eta \tag{4.45}$$

where $A(\tau)$ and $B(\tau)$ are the amplitude functions and their explicit form can be determined by substituting (4.45) on the right hand side of eqn. (4.44) and suppressing secular terms. Following the procedure given in Chapter 2 suppression of the secular terms leads to

$$\begin{aligned} &\alpha \Delta A(\tau) + \frac{(1.16253a - \alpha)}{\mu} A(\tau) + \frac{(1.16253b - \beta)}{\mu} A(\tau) \\ &- \frac{0.285}{\mu} A^3(\tau)(a+b)^3 + \frac{0.03643}{\mu} A^5(\tau)\ (a+b)^5 - \frac{0.00167}{\mu} \\ &\qquad A^7(\tau)\ (a+b)^7 = 0 \end{aligned}$$

under steady state the above equation takes the following form

$$\begin{aligned} &1.16253 - 0.285\, A^2(a+b)^2 + 0.03643A^4\ (a+b)^4 - 0.00167 \\ &\qquad A^6(a+b)^6 = \frac{1}{a+b} \end{aligned} \tag{4.46}$$

since $\alpha + \beta = 1.0$.

It is to be noted that under steady state situation the generating solution becomes

$$x_0(\eta,\tau) = A, \text{ a constant}$$

since $|\beta| < 1.0$ along the side AC of the stability triangle. For this reason it is not necessary to collect the secular terms corresponding to $(-\beta)^\eta$ term.

Then $y = A(a+b)$ with $|y| \leq 3.0$

with the above constraint the eqn. (4.46) gives the following region

$$0.98 < a + b < 3.02 .$$

Then the complete region in a-b plane in which P_1 limit cycle oscillations are possible is obtained by increasing D and is given by

$$a + b > 0.98 . \qquad (4.47)$$

Similar analysis can be carried out to obtain regions corresponding to P_2, P_3, P_4 etc. limit cycle oscillations by suitably selecting a_1 and a_2 in the basic system described in eqn. (4.36) so that the linear response is of periodic form with required period.

The various regions, showing P_L limit cycle oscillations are indicated in Fig. 4.6.

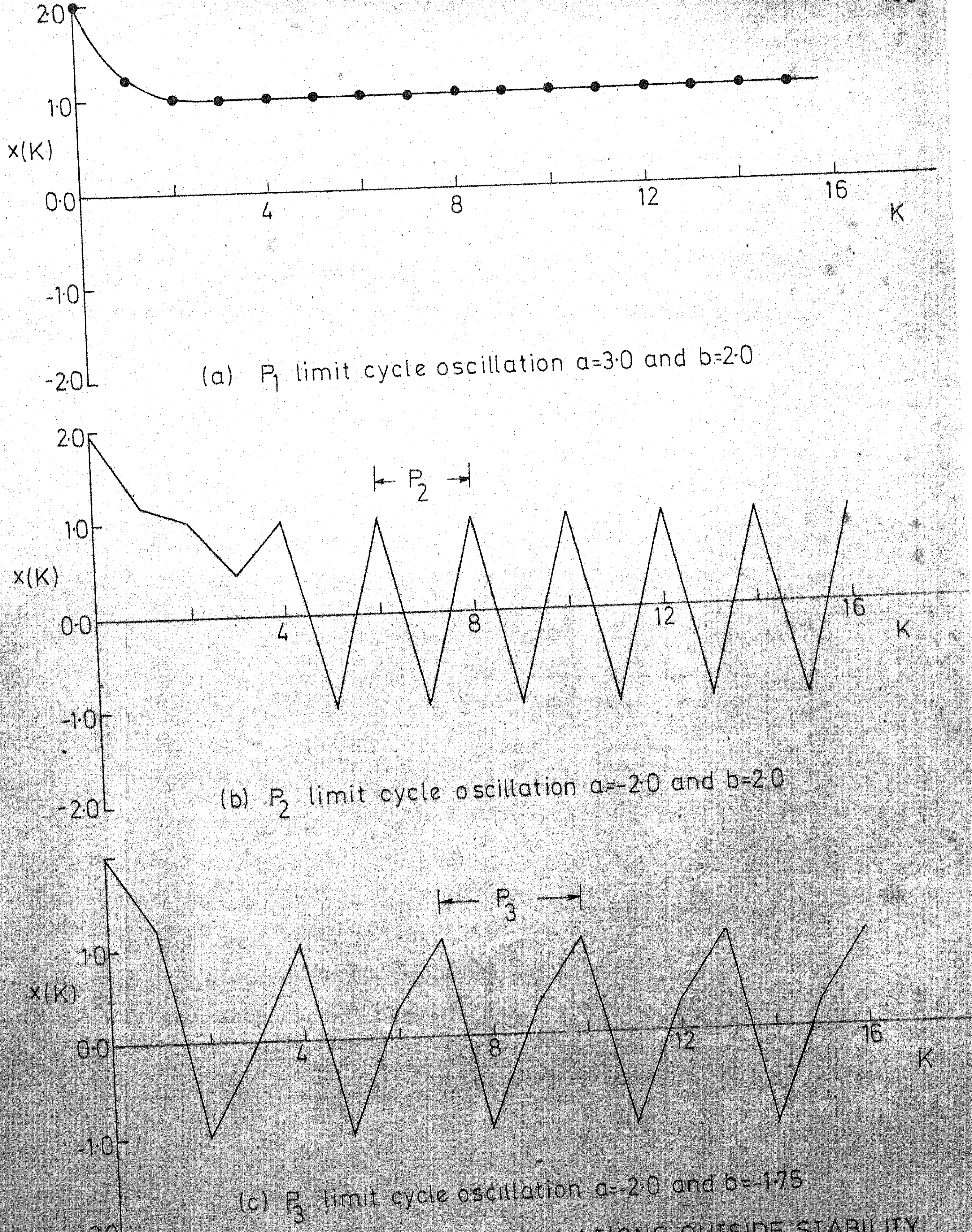
4.5 Simulation Results :

The theoretical findings given in the previous section are varified by simulating the given system as a recurrence relation. Fig. 4.6 shows different regions in the a-b parameter plane where limit cycle oscillation of different periods to exists. Figs. 4.7(a) - 4.7(f) show the time response for different set of [a,b] values taken from different regions shown in Fig. 4.6. Note that Fig. 4.7(f) shows P_{10} oscillation for a particular [a,b] values. This response changes considerably to another periodic response for small change in [a,b].

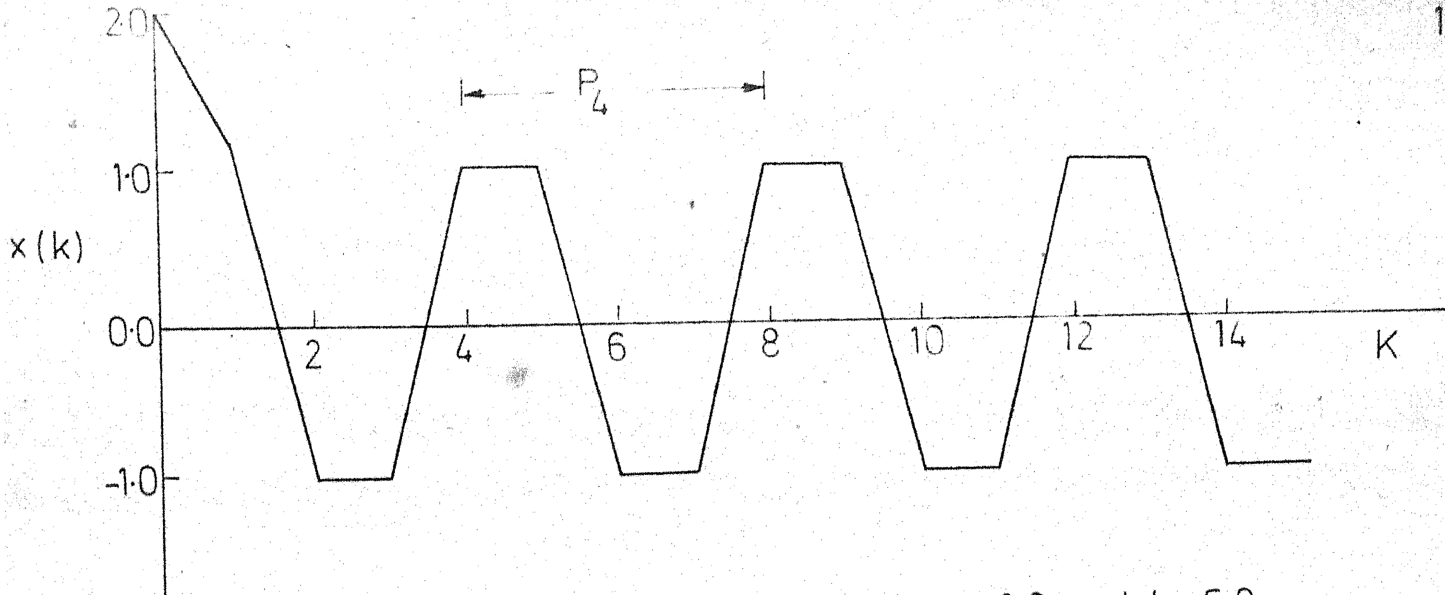
The stability aspects of limit cycle oscillations are discussed in the following section.

4.6 Stability of Limit Cycle Oscillations :

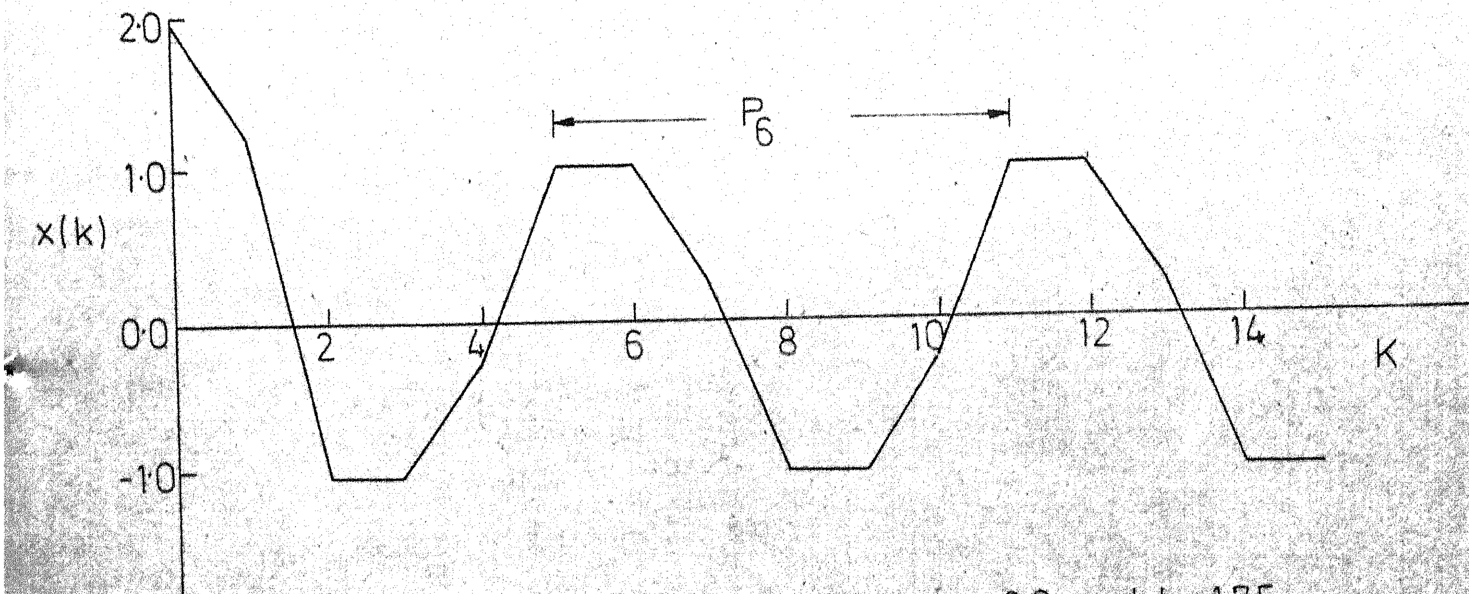
The simulated results obtained in the previous section show that the maximum magnitude of the limit cycle oscillation is unity irrespective of its period of oscillation. This is true due to the saturation nonlinearity which clamped the output at $\pm$ 1.0 whenever it exceeds unity. The stability of limit cycle oscillations can be studied through variation techniques. A small variation or perturbation is assumed around the known periodic limit cycle oscillation and a variational equation is obtained. This equation is used to study the stability of limit cycles whose output sequence is known



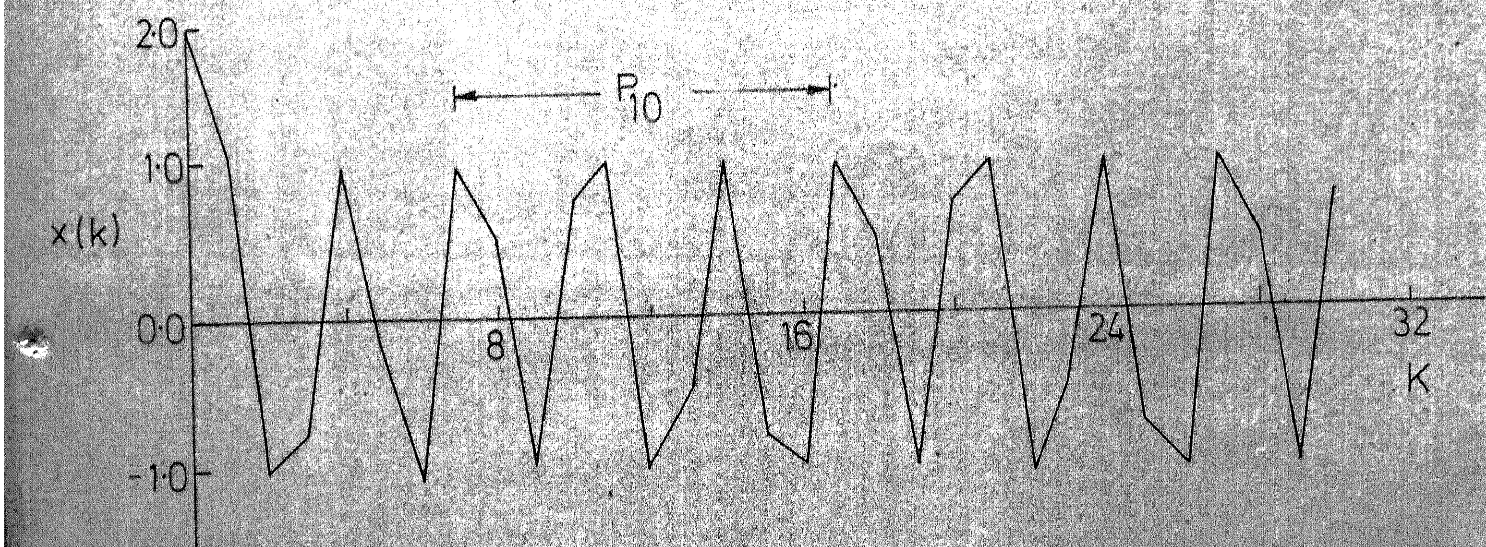
FIG·4·7 P_L LIMIT CYCLE OSCILLATIONS OUTSIDE STABILITY TRIANGLE



(d) P_4 limit cycle oscillation a= 2·0 and b=-5·0



(e) P_6 limit cycle oscillation a=2·0 and b=-1·75



(f) P_{10} limit cycle oscillation a=-2·0 and b=-2·5

FIG.4·7 P_i LIMIT CYCLE OSCILLATIONS OUT SIDE STABILITY TRIANGLE

beforehand. As mentioned in the introductory chapter the stability of a limit cycle falls into any one of the following categories.

(i) Stable limit cycle : A stable limit cycle is one for which the solution to variatiaval equation tends to zero as k increases to infinity.

(ii) Semistable limit cycle : A limit cycle is known to be semistable if the variational equation exhibits directional stability properties.

(iii) unstable limit cycle : The limit cycle oscillation is unstable if the variational equation gives an unbounded response.

The variational equation for stability study of limit cycles in digital filters is obtained as follows :

Let $x^*(k)$ be a known sequence representing a known limit cycle oscillation. Then introducing a small variation about $x^*(k)$ the filter variable $x(k)$ can be expressed as

$$x(k) = x^*(k) \pm \delta x(k)$$

where $\delta x(k)$ is small perturbation about the limit cycle oscillation. Substituting this in filter equation (4.1) we obtained.

$$x^*(k+2) \pm \delta x(k+2) = f[ax^*(k+1) + bx^*(k) \pm \delta x(k+1) \pm \delta x(k)]$$

from which, for a positive valued perturbation

$$\delta x(k+2) = f[ax^*(k+1) + bx^*(k) + a\delta x(k+1) + b\delta x(k)] - x^*(k+2) \tag{4.48}$$

and for a negative valued perturbation

$$\delta x(k+2) = -f[ax^*(k+1) + bx^*(k) - a\delta x(k+1) - b\delta x(k)] + x^*(k+2). \tag{4.49}$$

The equations (4.48) and (4.49) can be used to study the stability of given limit cycle oscillation. As mentioned in the beginning, if $\delta x(k) \to 0$ as $k \to \infty$ in both the equations the limit cycle is a stable limit cycle, if $\delta x(k) \to M$, as $k \to \infty$, where M is finite, then the limit cycle is unstable, whereas if $\delta x(k) \to 0$ as $k \to \infty$ in one equation and $\delta x(k) \to M$ as $k \to \infty$ in the other equation then the limit cycle is said to be semistable.

For parameter values outside the stability triangle the linear response is unstable for all initial conditions. This kind of unstable response causes limit cycle oscillations in nonlinear case with saturation type of nonlinearity for almost all sets of [a,b] values, except at certain points which will be discussed later on in the analysis. The simulation result of the filter eqn. (4.1) shows that whenever there is a limit cycle oscillation for the parameter values outside the stability triangle, the limit cycle is always stable and this property can be checked by considering various examples.

For convenience and easy reference the P_L limit cycle oscillations given in Fig. 4.7 are considered for the stability

(i) Fig. 4.7(a) gives P_1 limit cycle with following sequence

$$x^*(o) = 2.0$$
$$x^*(1) = 1.2$$
$$x^*(2) = x^*(3) = \ldots\ldots = 1.0$$

considering the initial disturbances $\delta x(o) = 0.1$, $\delta x(1) = 0.1$ the following values can be obtained for $\delta x(k)$ in the eqn. (4.48)

$$\delta x(2) = f[3.6 + 4.0+0.3 + 0.2] - 1.0 = 0.0$$

$$\delta x(3) = f[3.0+2.4+0.0 + 0.2] - 1.0 = 0.0$$

and

$$\delta x(k) = 0.0 \;\forall\, k \geq 4.$$

which shows $\delta x(k) \to o$ as $k \to \infty$.

and similarly from the eqn. (4.49)

$$\delta x(2) = -f[3.6 + 4.0 - 0.3 - 0.2] + 1.0 = 0.0$$
$$\delta x(3) = -f[3.0+2.4+0.0 - 0.2] + 1.0 = 0.0$$

and $\delta x(k) = 0 \quad \forall \quad k \geq 4$

this also shows $\delta x(k) \to o$ as $k \to \infty$. Therefore the limit cycle is a stable limit cycle.

(ii) P_2 limit cycle is sketched in Fig. 4.7(b) and its sequence is

$$x^*(o) = 2.0$$

$$x^*(1) = 1.2$$

$$x^*(2) = 1.0$$

$$x^*(3) = 0.4$$

$$x^*(4) = 1.0$$

$$x^*(5) = -1.0$$

and
$$x^*(k) = 1.0 \quad \text{for } k \text{ even}$$
$$= -1.0 \text{ for } k \text{ odd}$$

Then with $\delta x(o) = 0.1$ and $\delta x(1) = 0.1$ and using eqn. (4.48) we obtain

$$\delta x(2) = f[-2.4 + 4 - 0.2 + 0.2] - 1.0 = 0.0$$

$$\delta x(3) = f[-2 + 2.4 + 0.2] - 0.4 \qquad = 0.2$$

$$\delta x(4) = f[-0.8 + 2.0 - 0.4] - 1.0 \qquad = -0.2$$

$$\delta x(5) = f[-2 + 0.8 + 0.4 + 0.4] + 1.0 = 0.6$$

$$\delta x(6) = f[2.0 + 2.0 - 1.2 - 0.4] - 1.0 = 0.0$$

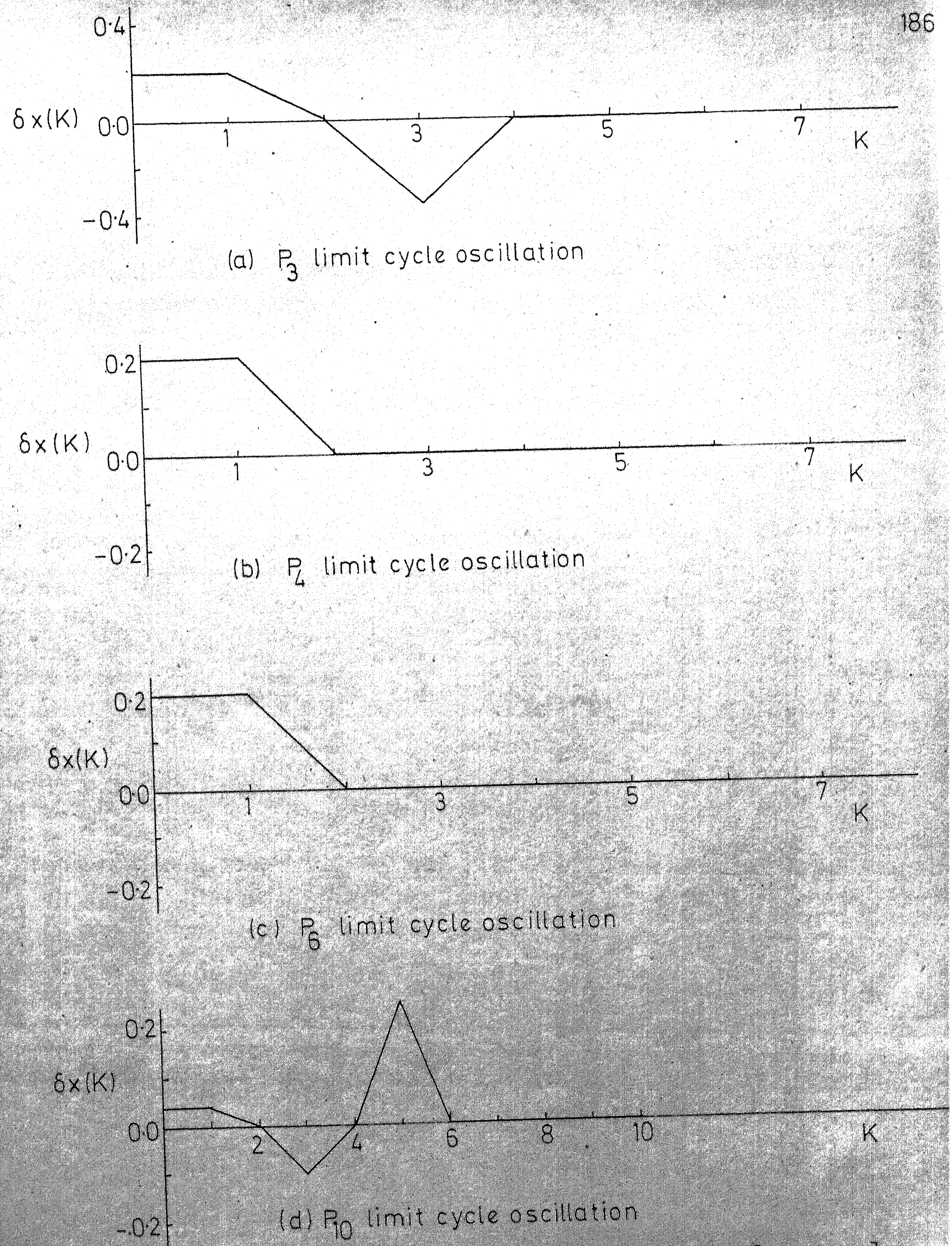
$$\delta x(7) = f[-2.0 - 2.0 + 0.0 + 1.2] + 1.0 = 0.0$$

$$\delta x(8) = \delta x(9) = \ldots . = 0.0$$

that is $\delta x(k) \to o \quad$ as $k \to \infty$

and a similar result can be obtained with eqn. (4.49). This shows the limit cycle oscillation shown in Fig. 4.7(b) is stable.

In a similar way the sequence $\delta x(k)$ for other periods of limit cycles are obtained and plotted in Figs. 4.8 and 4.9. Fig. 4.8 is the time response of the variational equation (4.48) and Fig. 4.9 is that of (4.49). The plots show the



FIG·4·8 STABILITY OF LIMIT CYCLES IN FIG. 4·7 [EQ.(4·48)]

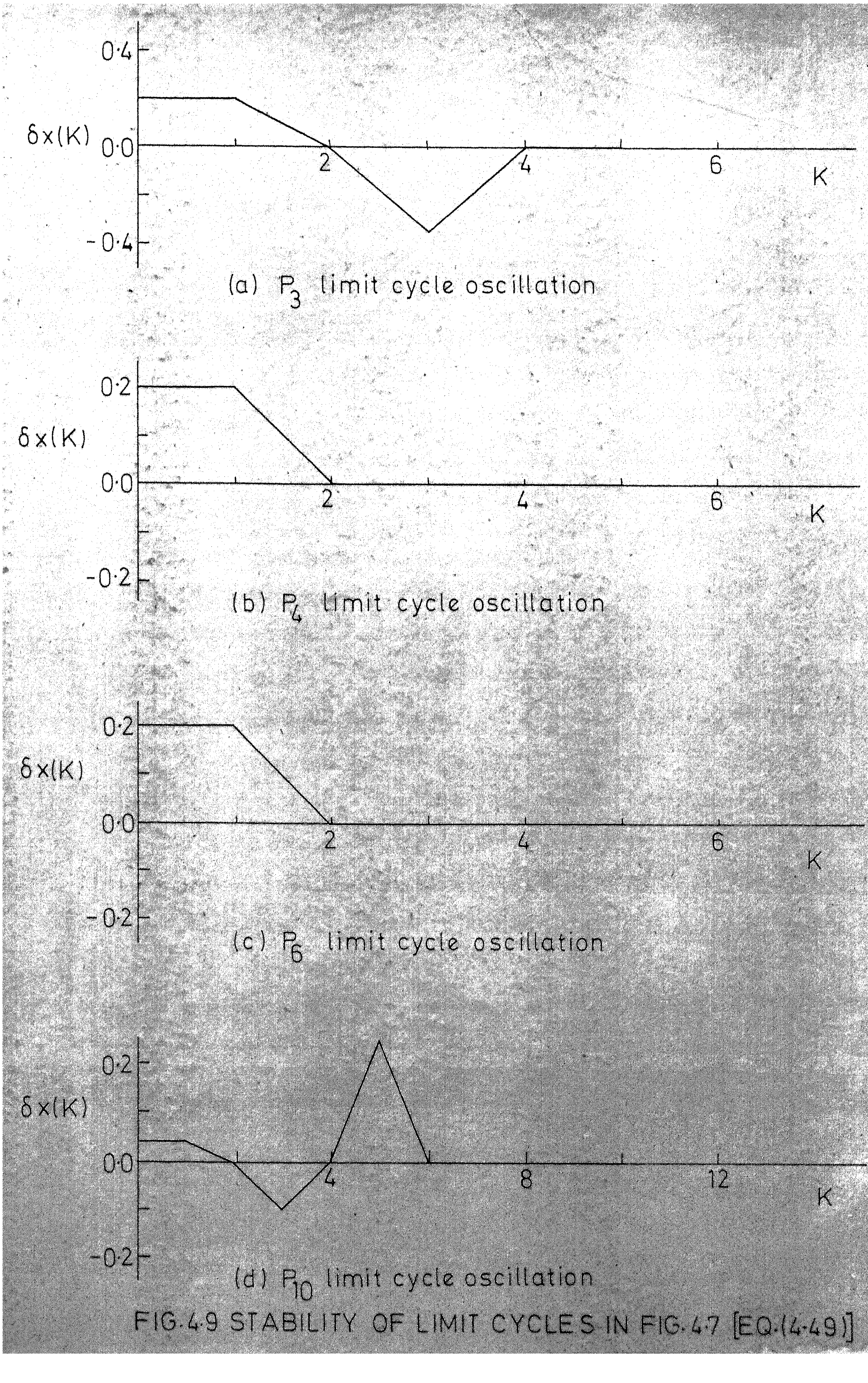


FIG.4.9 STABILITY OF LIMIT CYCLES IN FIG.4.7 [EQ.(4.49)]

limit cycle oscillations are stable.

The dependence of the limit cycle oscillations (for parameter values inside the stability triangle) on the initial condition has been briefly remarked by Claasen et al [79] and Willson [91]. The limit cycles outside the stability triangle are also seen to be dependent on the initial conditions. This shows that the term 'limit cycles' in digital filters has a different meaning than that for nonlinear continuous time system.

It is observed that, due to the instability behaviour of the linear response for the parameter values outside the stability triangle, the assumption that a limit cycle oscillation is always possible is not true with saturation nonlinearity. For given parameter values [a,b] outside the stability triangle there exists a set of initial conditions [x(o), x(1)] such that the nonlinear response dies down to origin monotonically. In otherwords for particular filter coefficients outside the stability triangle the linear response is unstable whereas the nonlinear response with saturation nonlinearity is asymptotically stable. This interesting phenomenon is explained in the following section.

4.7 Monotonic Decaying Response Outside the Stability Triangle :

It is well known that the response to a linear second order difference equation is asymptotically stable for the parameter

[a,b] values within the stability triangle, generally bounded when they are on the sides of the triangle and unstable for their location outside the triangle. Once the saturation nonlinearity is considered the response to a second order digital filter, for some [a,b] values located inside the stability triangle, becomes unstable and leads to self sustained oscillations [76, 79, 83]. This interesting investigation tempts one to investigate the possible existence of asymptotically stable response for the parameter values outside the stability triangle with saturation arithmetic. The details of the above investigations are given below :

4.7.1 Analysis :

For a given a and b, a set of initial values [x(o), x(1)] is obtained for which the response is asymptotically stable. Consider x(1) related to x(o) as

$$x(1) = m\, x(o)$$

where

$$|m| \leq 1.0$$

Then

$$x(2) = f[ax(1) + bx(o)]$$
$$= f[(am+b)\, x(o)]$$

Now x(2) can never exceed unity in absolute value, that is

$|x(2)| = 1.0$ for $|(am+b)\, x(o)| > 1.0$

and $x(2) = (am + b)\, x(o)$ otherwise.

$$x(3) = f[(a^2m + ab + bm)\, x(o)]$$
$$= [(a^2 + b)\, m + ab]\, x(o)$$

because of the assumption of monotonic decay.

$$x(4) = [(a^3 + 2ab)\, m + a^2b + b^2]\, x(o)$$

.....

.....

.....

$$x(N) = [(\alpha_1 a^{N-1} + \alpha_2 a^{N-3} b + \alpha_3 a^{N-5}\, b^2 + \ldots.)m$$
$$+ (\beta_1 a^{N-2} b + \beta_2\, a^{N-4}\, b^2 + \beta_3 a^{N-6} b^3 + \ldots)]x(o$$

Now, when N is sufficiently large

$$x(N) = 0.0$$

and the slope m can be evaluated from the right hand side of the equation for x(N) as :

$$m = \frac{-[\beta_1 a^{N-2} b + \beta_2 a^{N-4} b^2 + \beta_3 a^{N-6} b^3 + \ldots.]}{[\alpha_1 a^{N-1} + \alpha_2 a^{N-3} b + \alpha_3 a^{N-5} b^2 + \ldots\ldots]} \qquad (4.50)$$

where α_i and β_i are the coefficients and are given in table 4.1 and 4.2 for different N.

The elements in tables 4.1 and 4.2 are obtained as follows

Let

α_{i1} ($i = 1,2,3, \ldots$) be the elements in the first column. Then the Table 4.1 can be filled-up with $a_{i1} = 1.0\,\forall i$ and

Table 4.1

Values of α_i for different N

$N \backslash \alpha_i$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
1	1									
2	1									
3	1									
4	1	2								
5	1	3	1							
6	1	4	3							
7	1	5	6	1						
8	1	6	10	4						
9	1	7	15	10	1					
10	1	8	21	20	5					
11	1	9	28	35	15	1				
12	1	10	36	56	35	6				
13	1	11	45	84	70	21	1			
14	1	12	55	120	126	56	7			
15	1	13	66	165	210	126	28			
16	1	14	78	220	330	252	84	8		
17	1	15	91	286	495	462	210	36	1	
18	1	16	105	364	715	792	462	120	9	
19	1	17	120	455	1001	1287	924	330	45	1
20	1	18	136	560	1365	2002	1716	792	165	10

Table 4.2

Values of β_i for different N

N \ β_i	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
1	0									
2	1									
3	1	0								
4	1	1								
5	1	2	0							
6	1	3	1							
7	1	4	3	0						
8	1	5	6	1						
9	1	6	10	4	0					
10	1	7	15	10	1					
11	1	8	21	20	5	0				
12	1	9	28	35	15	1				
13	1	10	36	56	35	6	0			
14	1	11	45	84	70	21	1			
15	1	12	55	120	126	56	7	0		
16	1	13	66	165	210	126	28	1		
17	1	14	78	220	330	252	84	8	0	
18	1	15	91	286	495	462	210	36	1	
19	1	16	105	364	715	792	462	120	9	0
20	1	17	120	455	1001	1287	924	330	45	1

and using the relationship :

$$a_{N+2(j-1),j} = \sum_{k=1}^{N} a_{k+2(j-2),j-1}, \quad N=1,2,3, \ldots,$$
$$j = 2,3,4, \ldots$$

where j stands for column.

In the same way elements of table 4.2 can also be obtained with $a_{11} = 0.0$ and $a_{i1} = 1.0 \quad \forall\ i \geq 2$ and

$$a_{N+2(j-1),j} = \sum_{k=1}^{N} a_{k+2(j-2),j-1}, \quad N=1,2,3, \ldots,$$
$$j = 2,3,4, \ldots,$$

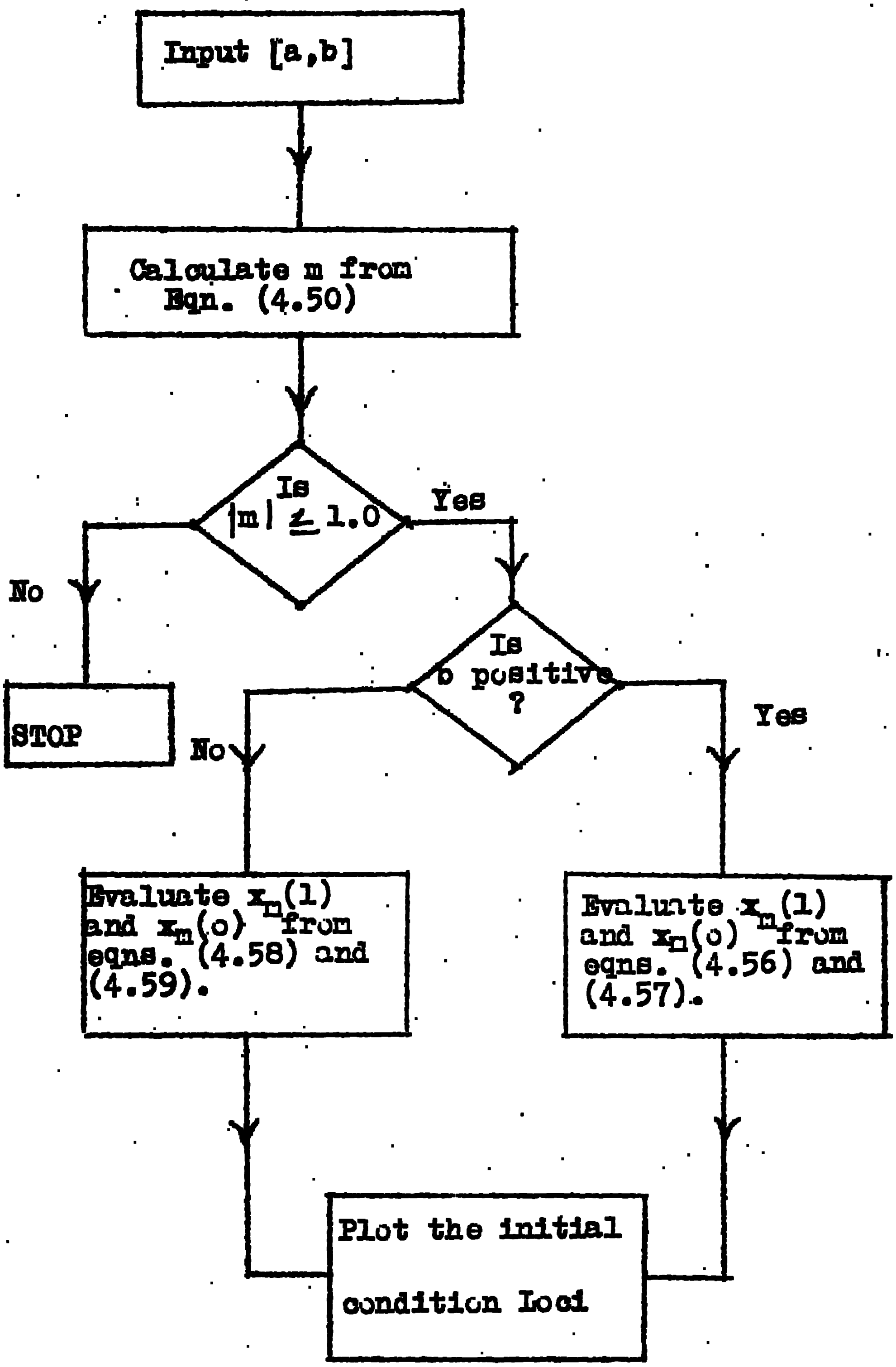
Before examining the detailed analysis, it is useful to outline the steps involved with the help of following flow charts.

The flow chart 4.1 explains the method of obtaining the initial condition locus when the [a,b] values are specified, whereas the flow chart 4.2 gives the steps to be followed to obtain a set of [a,b] values from a given [a,b] for monotonic decaying kind of response. The set of initial conditions for the different [a,b] values are just translation of the initial condition loci obtained from flow chart 4.1.

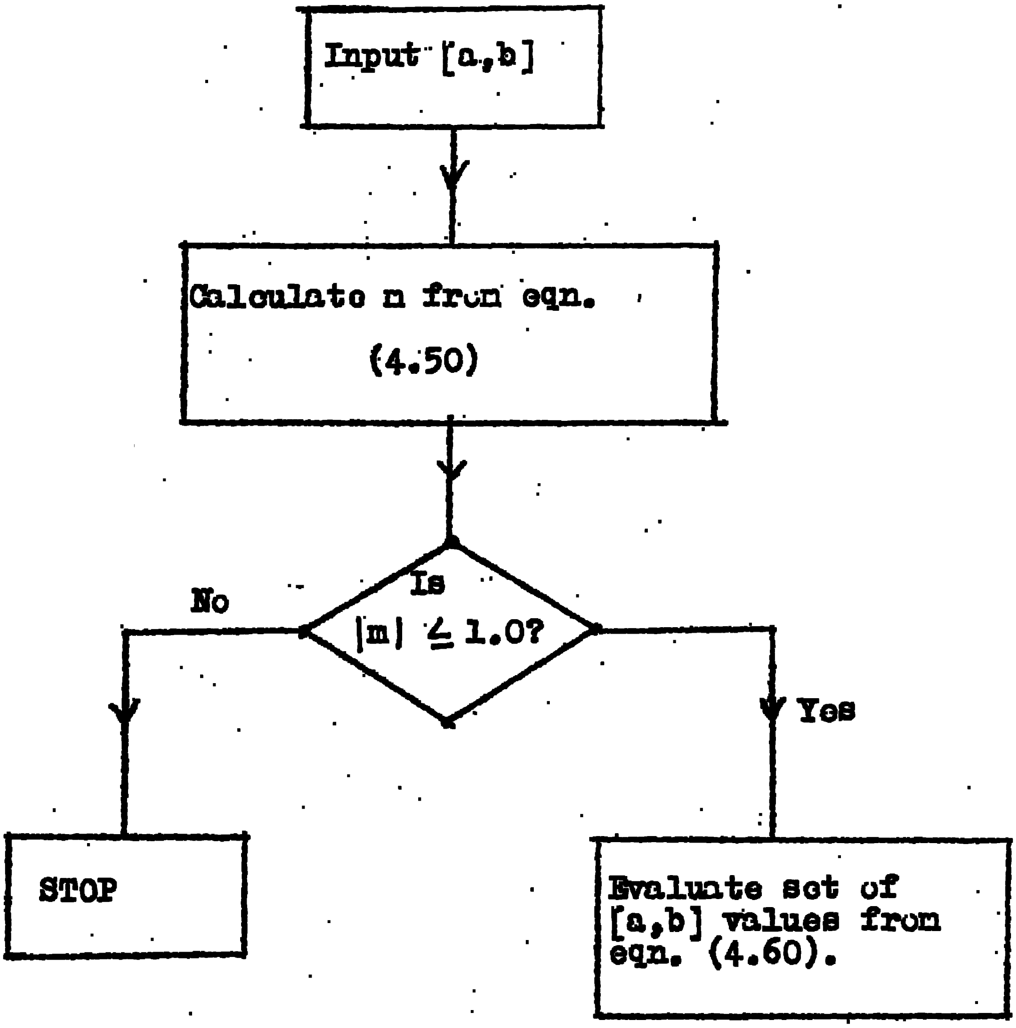
The value of m is known for a given [a,b] from the eqn. (4.50)

Then,

$$x(2) = \left(\frac{am+b}{m}\right) x(1) = (am+b)\, x(o).$$



Flow Chart 4.1 Initial condition Loci for Monotonic decay response



Flow Chart 4.2 Set of [a,b] values for monotonic decay response.

Now the maximum amplitude that $x(2)$ can assume is 1.0 then $x_m(1)$ the value of $x(1)$ corresponding to the maximum value of $x(2)$ is obtained as

$$1 = \left(\frac{am+b}{m}\right) x_m(1)$$

from which

$$x_m(1) = \frac{m}{am+b} \qquad (4.51)$$

Similarly

$$x_m(o) = \frac{1}{am+b} \qquad (4.52)$$

again from

$$x(2) = f[ax(1) + bx(o)]$$

$$\text{for } ax(1) + bx(o) > 1.0$$

$$x(2) = 1.0 \; .$$

Under this condition $\quad x(1) = x_m(1) = \frac{m}{am+b}$

that is

$$\frac{am}{am+b} + bx(o) > 1.0$$

$$bx(o) > \frac{b}{am+b} \; . \qquad (4.53)$$

Likewise for

$$ax(1) + bx(o) < -1.0$$

$$x(2) = -1.0$$

for this condition $\quad x(1) = x_m(1) = \frac{-m}{am+b}$

then $-bx(o) > \frac{b}{am+b}$ (4.54)

also when $|ax(1) + bx(o)| \leq 1.0$, (4.54)

$x(2) = ax(1) + bx(o)$.

this gives the fiter operation in the linear range of the saturation nonlinearity, that is

$$x(2) = \left(\frac{am+b}{m}\right) x(1)$$

which implies $x(1) = mx(o)$ (4.55)

<u>Case 1</u> b positive

from eqn. (4.53)

$$x(o) \geq \frac{1}{am+b} = x_m(o) \quad \text{and} \quad x_m(1) = \frac{m}{am+b} \qquad (4.56)$$

from eqn. (4.54)

$$x(o) \leq \frac{-1}{am+b} = -x_m(o) \quad \text{and} \quad x_m(1) = \frac{-m}{am+b} \qquad (4.57)$$

The conditions given in eqns. (4.55), (4.56) and (4.57) are sketched in Fig. 4.10 for m positive and in Fig. 4.11 for m negative.

<u>Case 2</u> b negative

from eqn. (4.53)

$$x(o) \leq \frac{1}{am+b} = x_m(o) \quad \text{and} \quad x_m(1) = \frac{m}{am+b} \qquad (4.58)$$

from eqn. (4.54)

$$x(o) \geq \frac{-1}{am+b} = -x_m(o) \quad \text{and} \quad x_m(1) = \frac{-m}{am+b} \qquad (4.59)$$

The above conditions along with the condition given in eqn. (4.55) are plotted in Figs. 4.12 and 4.13 for m positive and negative respectively.

The Figs. 4.10 - 4.13 give the set of initial values $[x(o), x(1)]$ for a given $[a,b]$ in the parameter plane for which the filter response decays to zero monotonically.* The initial condition loci in Figs. 4.10 - 4.13 give a monotonic decaying response not only for a specified $[a,b]$ value, but also for a set of $[a,b]$ values connected with slope m as given below :

$$x(2) = f[ax(1) + bx(o)]$$

when the filter is just nonlinear for

$$x(2) = 1.0 = ax(1) + bx(o)$$

or $$x(2) = -1.0 = ax(1) + bx(o)$$

also we know for

$$x(2) = 1.0, \quad x(1) = x_m(1) = \frac{m}{am+b}$$

and for

$$x(2) = -1.0, \quad x(1) = -x_m(1) = \frac{-m}{am+b}$$

then

$$1.0 = \left(\frac{am+b}{m}\right) x_m(1)$$

from which

$$b = -am + \frac{m}{x_m(1)} \; . \qquad (4.60)$$

---

* Whenever the selection of initial conditions are in the linear segment portion, then the linear as well as the nonlinear system.

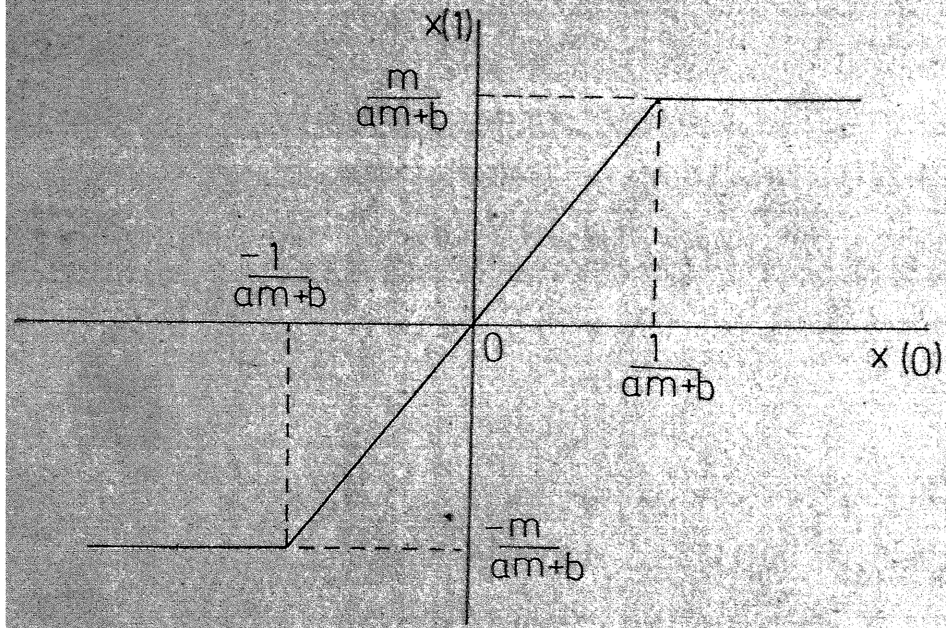


FIG. 4.10 (m Positive, b Positive)

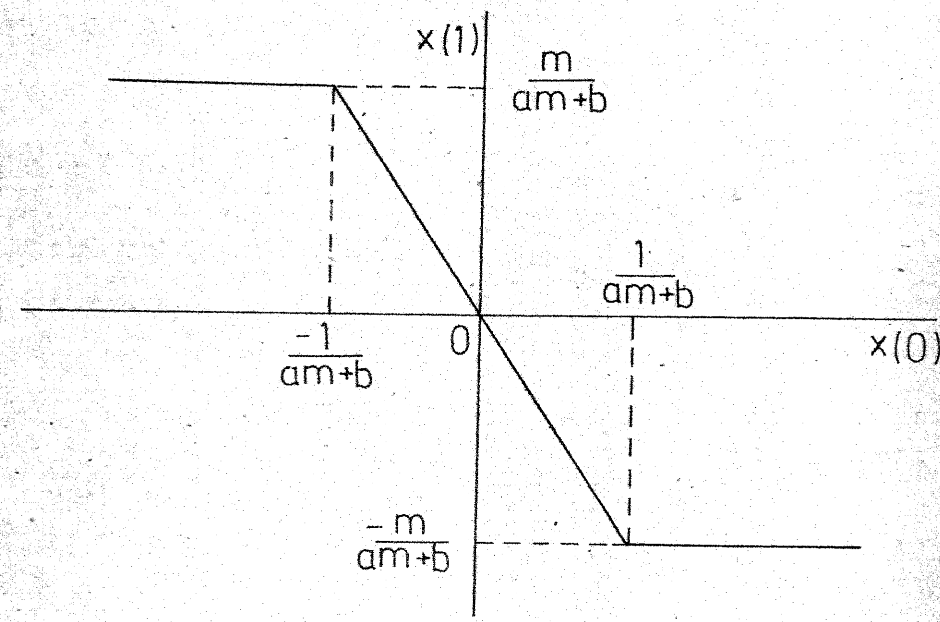


FIG. 4.11 (m Negative, b Positive)

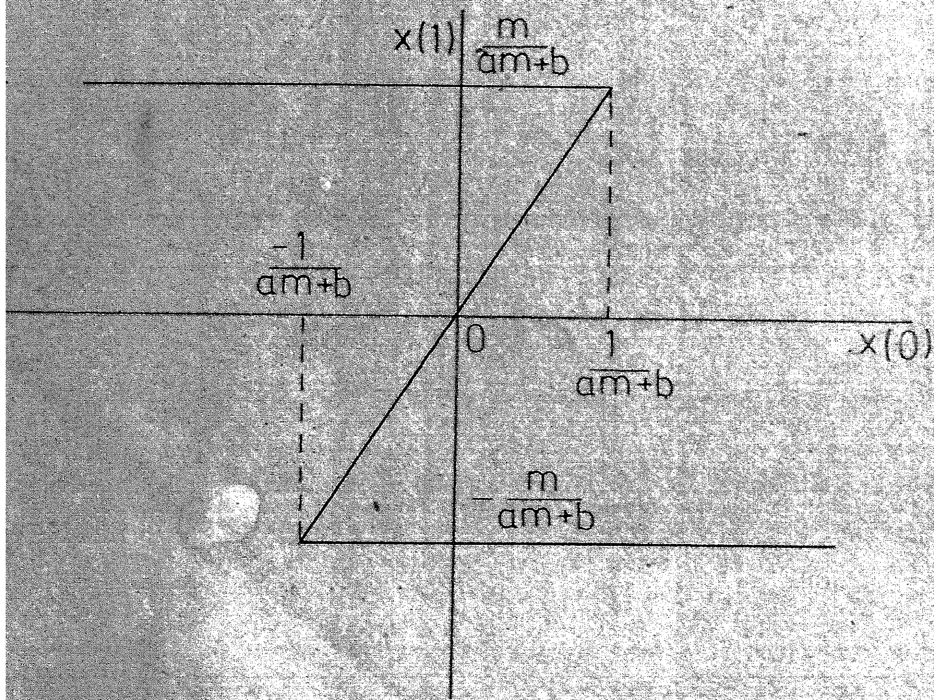


FIG. 4.12 (m Positive, b Negative)

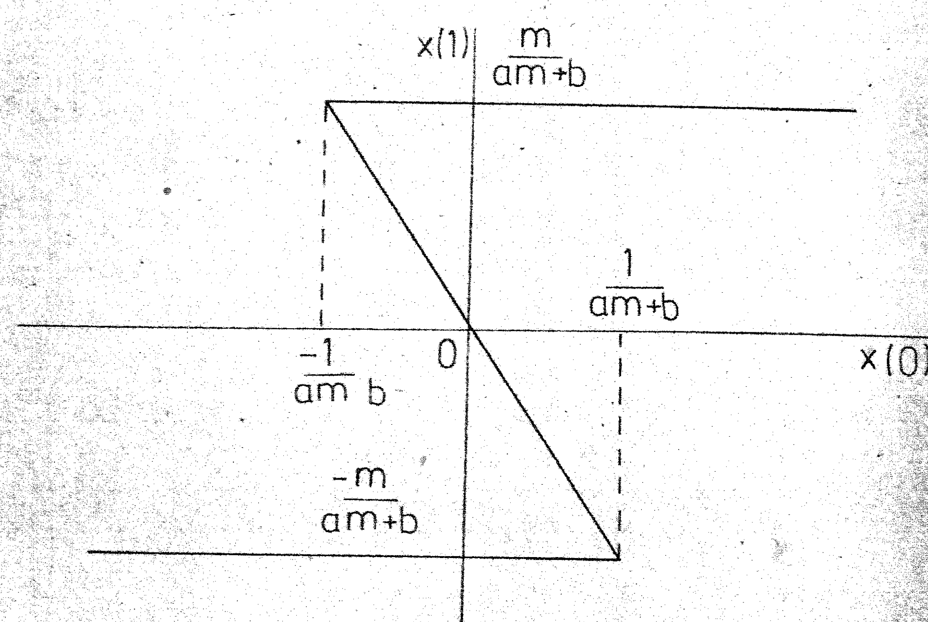


FIG. 4.13 (m Negative, b Negative)

Eqn. (4.60) gives a set of values [a,b] for which there exist a set of initial conditions [x(o), x(1)] as indicated in Figs. 4.10 - 4.13 for which the nonlinear response of the filter decays down to origin monotonically. The region in the a-b parameter plane in which such type of monotonic decaying response is possible is obtained as follows :

$$x(2) = ax(1) + bx(o)$$

$$x(1) = mx(o).$$

Dividing one by the other

$$\frac{x(2)}{x(1)} = \frac{ax(1) + bx(o)}{mx(o)} = \frac{am+b}{m}$$

then

$$\frac{x(2)}{x(1)} \leq 1 \text{ from the assumption}$$

that is

$$\frac{am+b}{m} \leq 1.0$$

m may be positive or negative, but $|m| \leq 1.0$.

For m positive

$$am+b < m$$

$$b < (1-a)\, m$$

$$\frac{b}{1-a} < m < 1.0$$

$$\frac{b}{1-a} < 1.0 \qquad (4.61)$$

For m negative

$$am + b < -m$$

$$b < -(1+a)m$$

$$\frac{b}{1+a} < -m < 1.0 . \qquad (4.62)$$

Eqns. (4.61) and (4.62) give a region in a-b plane in which the solution decays monotonically to the origin. This region is shown in Fig. 4.14.

Note that the above results are not in conflict with earlier discussed limit cycles since only for a particular set of initial conditions determined by the Figs. 4.10 - 4.13 will there arise a monotonic decaying sequence. In general the family of limit cycles is much larger than that of the decaying sequence for a specified [a,b]. It is interesting to note that there exist regions outside the stability triangle where there cannot arise any monotonically decreasing response.

The above discussed phenomena have been exactly verified by digital computer simulation and few results are given below.

4.8 Examples :

(i) For a = -1.5 and b = 1.0

Using eqn. (4.50), the value of m is obtained as 0.5. For this [a,b], the set of initial conditions that are responsible for monotonic decaying response is given in Fig. 4.15. Whereas the Figs. 4.16(a) - 4.16(d) show the time

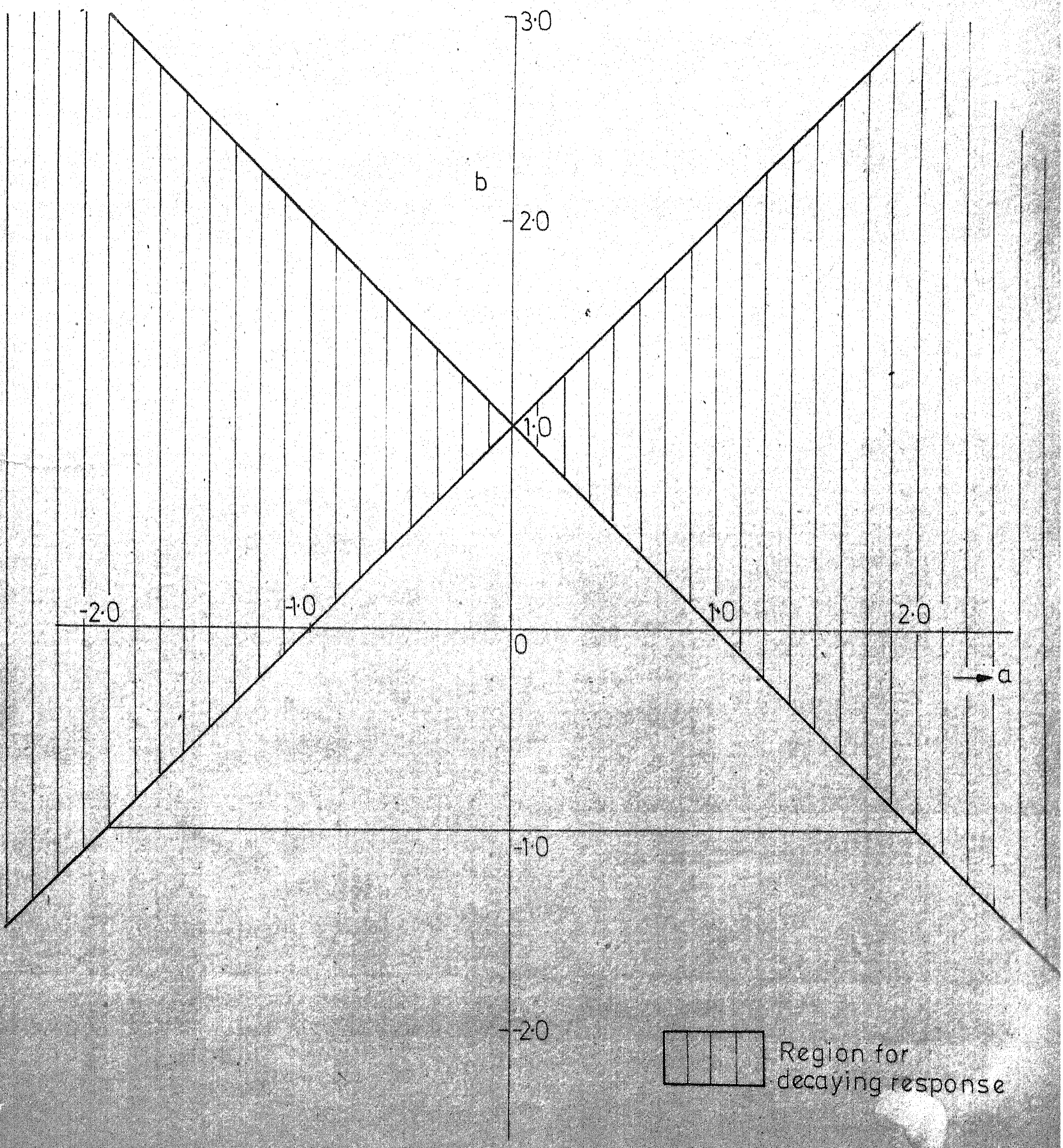


FIG.4.14 REGION IN a-b PLANE FOR DECAY RESPONSE

response of the system for different initial values [x(o), x(1)] taken from Fig. 4.15, that is

Fig. 4.16(a), for $x(1) = -2.0$ and $x(o) \leq -4.0$

Fig. 4.16(b), for $x(1) = 2.0$ and $x(o) \geq 4.0$

Fig. 4.16(c), for $x(1) = -1.5$ and $x(o) = -3.0$

Fig. 4.16(d), for $x(1) = 1.0$ and $x(o) = 2.0$

(ii) For $a = -2.0$ and $b = -0.75$

the value of m is obtained as -0.5

for this [a, b] value the initial condition locus is sketched in Fig. 4.17.

The time response for various chosen initial conditions from the Fig. 4.17 is shown in Fig. 4.18, that is

Fig. 4.18(a), $x(o) \geq -4.0$ and $x(1) = 2.0$

Fig. 4.18(b), $x(o) \leq 4.0$ and $x(1) = -2.0$

Fig. 4.18(c), $x(o) = 2.0$ and $x(1) = -1.0$

Fig. 4.18(d), $x(o) = -2.0$ and $x(1) = 1.0$

Thus the theoretical findings are exactly verified with simulated results of the given system.

4.9 Conclusion :

The existence of limit cycle oscillation in a second order digital filter, with overflow nonlinearity of saturation kind, is demonstrated with the parameter values outside the stability triangle in the a-b parameter space. Three methods have been proposed for locating regions in the parameter plane

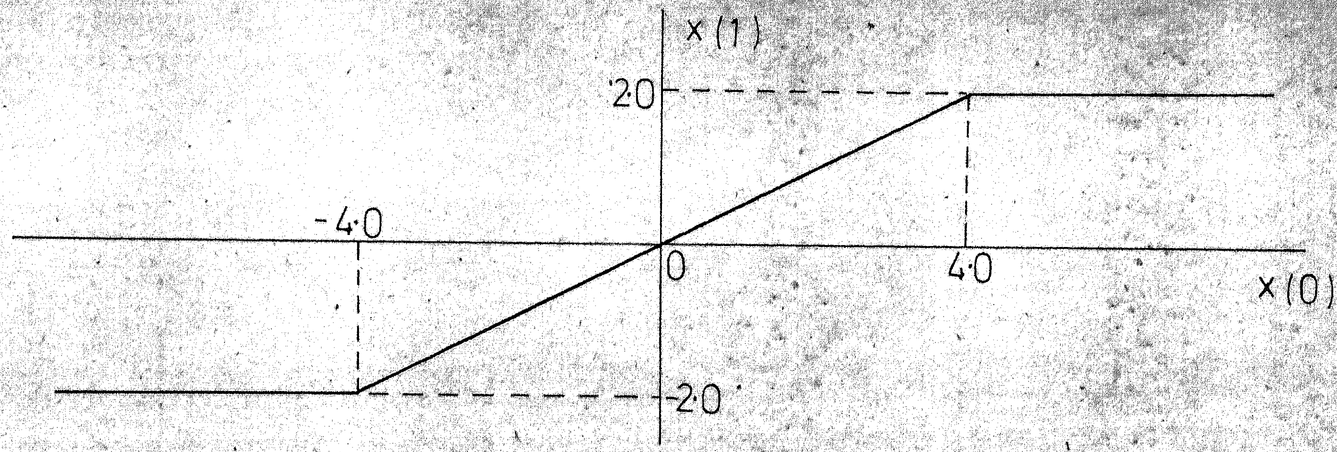
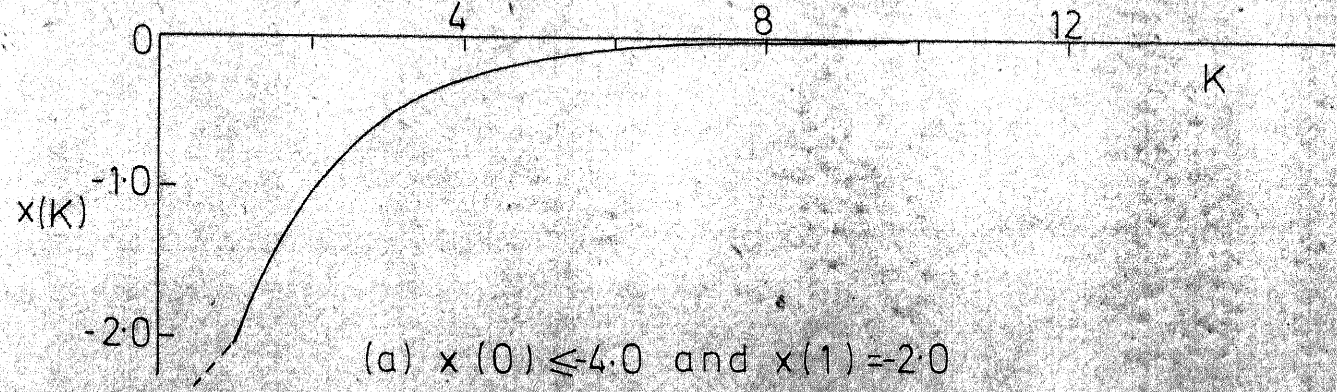
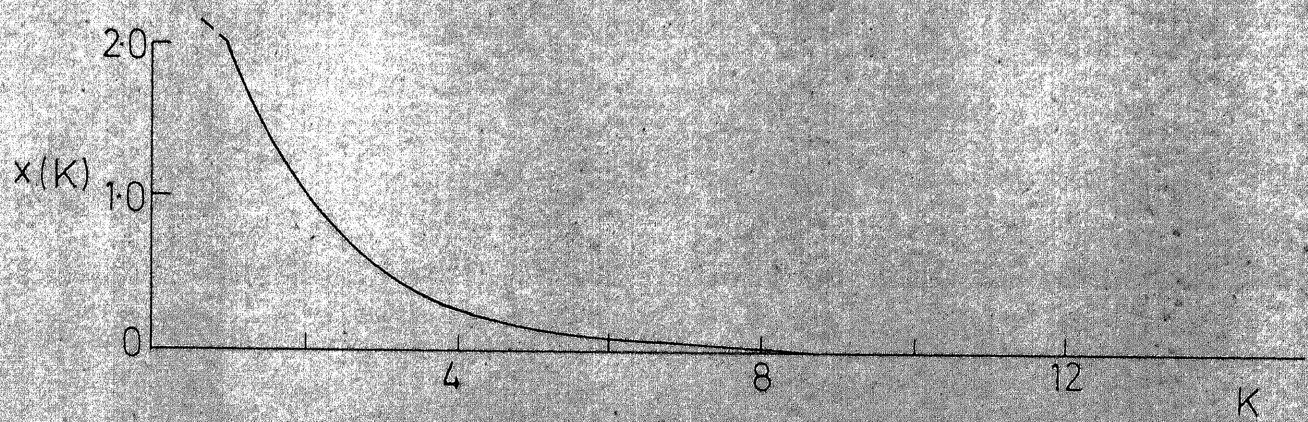


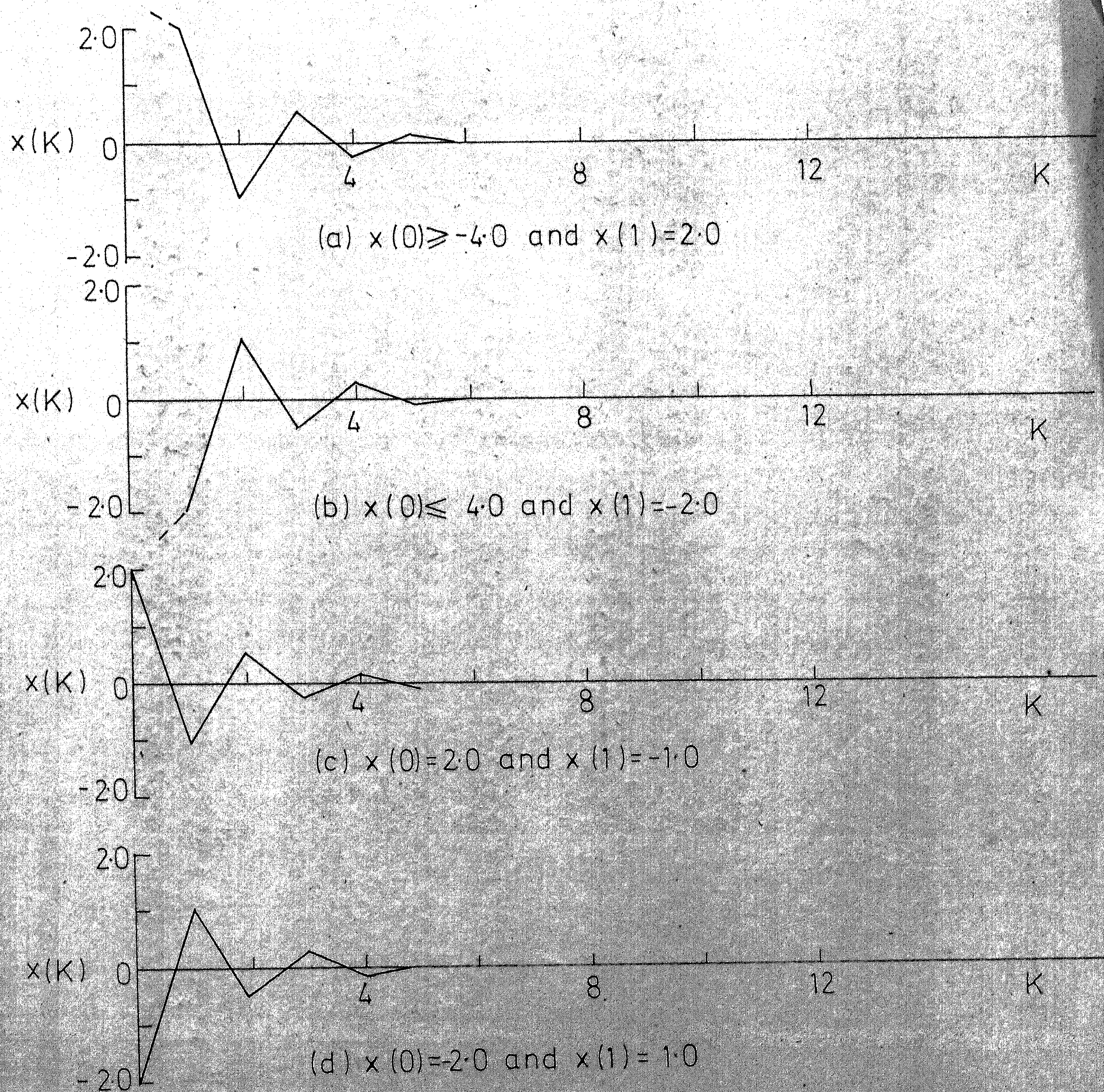
FIG. 4·15 INITIAL CONDITION LOCUS FOR a=-1·5, b=1·0 AND m=0·5



(a) x(0) ≤ -4·0 and x(1) = -2·0



(b) x(0) ≥ 4·0 and x(1) = 2·0



FIG·4·18 TIME RESPONSE OF THE FILTER FOR x(0) AND x(1) SELECTED FROM FIG·4·17

where there existslimit cycle oscillation of a specified period. The first of these is a variation of one proposed by Ebert et al [76] and the other two are based on a polynomial approximation to the saturation nonlinearity. In these two methods a lower degree polynomial with small range for the dependent variable may be considered and then by induction one can obtain the complete region where limit cycle oscillation of specified period is possible. The third method exploits the use of multiple scale perturbational approach and is believed to be quite novel.

Regions in the parameter plane where the output sequence decays monotonically to zero have also been ascertained. These are functions of the initial conditions and an explicit relation linking the parameter [a,b] and the initial data has been determined for this purpose. These relations both in the parameter space and in the initial condition space have been appropriately mapped.

CHAPTER 5

JUMP PHENOMENON IN DIGITAL FILTERS

5.1 Introduction :

The preceeding chapter has considered in detail the analysis of second order digital filters with saturation overflow nonlinearity under force free situation. The overflow nonlinear phenomenon introduced in the normal operations of the filter gives rise to limit cycle oscillations of different periods for parameter values outside the stability triangle. The stability analysis of such limit cycles was carried out using a variational technique. Further investigation had also revealed the existence of a region in the a-b parameter space outside the stability triangle where the output sequence decays down to zero monotonically with time. In this chapter a second order digital filter with an external input is considered. If an input signal is applied to the filter then the situation is different and more complicated than for the force free situation. For periodic input signals the quantization errors will be periodic too, and are sometimes referred to as limit cycles [93] and the nature of such limit cycles will be highly dependent on the specific input signal and the quantization effects are therefore not considered in this analysis. As mentioned earlier, the overflow though occuring

rarely in a properly designed digital filters introduces a large error in the normal operation of the filter and are the therefore investigated further in this chapter. The study of overflow oscillations in the forced response of a second order digital filter was initially due to Willson [94], who showed that unlike the zero input case, the response, for some choice of filter coefficients [a,b], differs from the ideal linear response and the filter turns out to be asymptotically unstable. The region in the a-b plane where this instability arises is shown in Fig. 2 of [94]. This result has recently been significantly extended by Claasen and Kristiansson [112], The following section is concerned with the jump phenomena in a forced second order digital filter with overflow saturation nonlinearity. This jump phenomenon will be shown to be closely related to the overflow nonlinear oscillation described in [94, 112].

5.2 Jump Phenomenon in Digital Filters :

A particular kind of nonlinear effect known as the jump phenomenon has been examined recently by Kristiansson [95] and Claasen and Kristiansson [96]. A somewhat related study is due to Willson [91] who has obtained different regions inside the stability triangle where forced overflow oscillations of different periods can be sustained. There appears to be no other specific study of this jump phenomenon in digital filters in the available literature.

Jump phenomena are characterised by changes in the steady state amplitude of the filter response that may occur due to small change in the amplitude or frequency of the periodic input signal. The present study is concerned with further investigations of the jump phenomena in a second order digital filter with a saturation nonlinearity arising due to overflow in the adder. The study provides information about the range of the input amplitude for which there exists a jump. For given values of the filter coefficients $[a, b]$ and input frequency α, conditions are derived which guarantee a jump or sustained nonlinear oscillations. A graphical construction of the derived conditions provides the location of the regions inside the stability triangle where such jump behaviour and nonlinear sustained oscillations are possible.

5.3 System Model :

The second order section of the digital filter under study is given in Fig. 5.1 and is described by

$$x(k+2) = f(ax(k+1) + bx(k) + z(k)) \tag{5.1}$$

where $f(\cdot)$ is the nonlinear function defining the saturation arithmetic by

$$f(\sigma) = \begin{cases} \sigma & \text{for } |\sigma| \leq 1.0 \\ \operatorname{sgn}(\sigma) & \text{for } |\sigma| > 1.0 \end{cases}$$

and the graphical representation is shown in Fig. 5.2. $Z(k)$ is the external input to the filter, generally periodic and is given by

$$Z(k) = F \cos \alpha k$$

where

F is the input amplitude and α is the input frequency.

Of interest is the examination of conditions under which disturbances (equivalent to small changes) in amplitude of input at specified times lead to a jump from one steady state (assumed periodic) solution to another steady state solution but with a different amplitude . Such an analysis is carried out in the following section.

5.4 Analysis :

The linear version of the nonlinear system described in eqn. (5.1) is

$$x(k+2) = ax(k+1) + bx(k) + Z(k) \tag{5.2}$$

whose solution is

$$x(k) = x_t(k) + x_s(k) \tag{5.3}$$

where

$x_t(k)$ is the transient solution obtained assuming zero input

$x_s(k)$ is the contribution due to the forcing function.

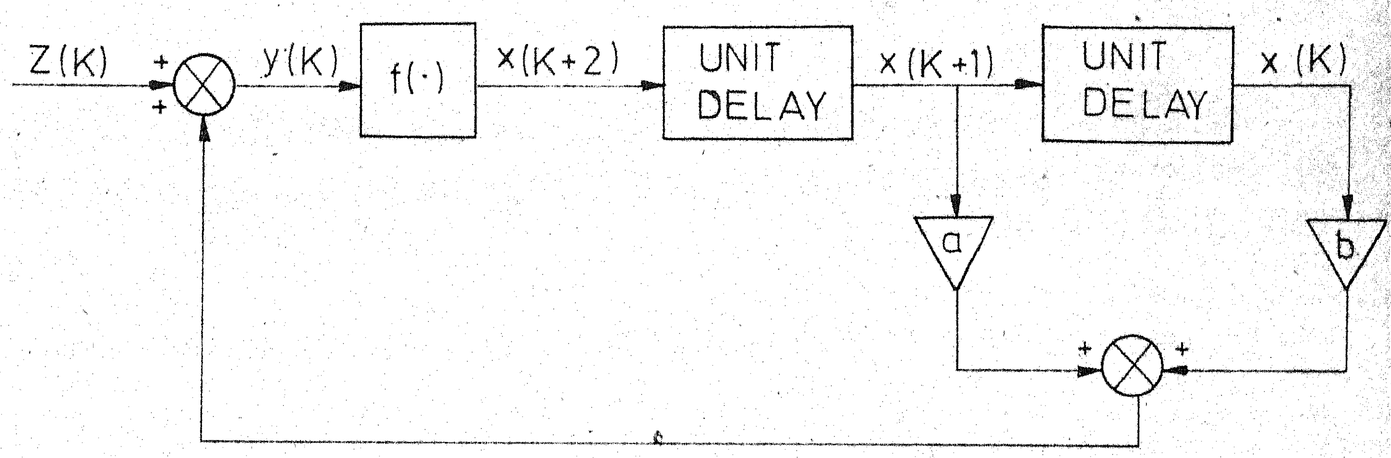


FIG.5.1 FORCED SECOND ORDER DIGITAL FILTER

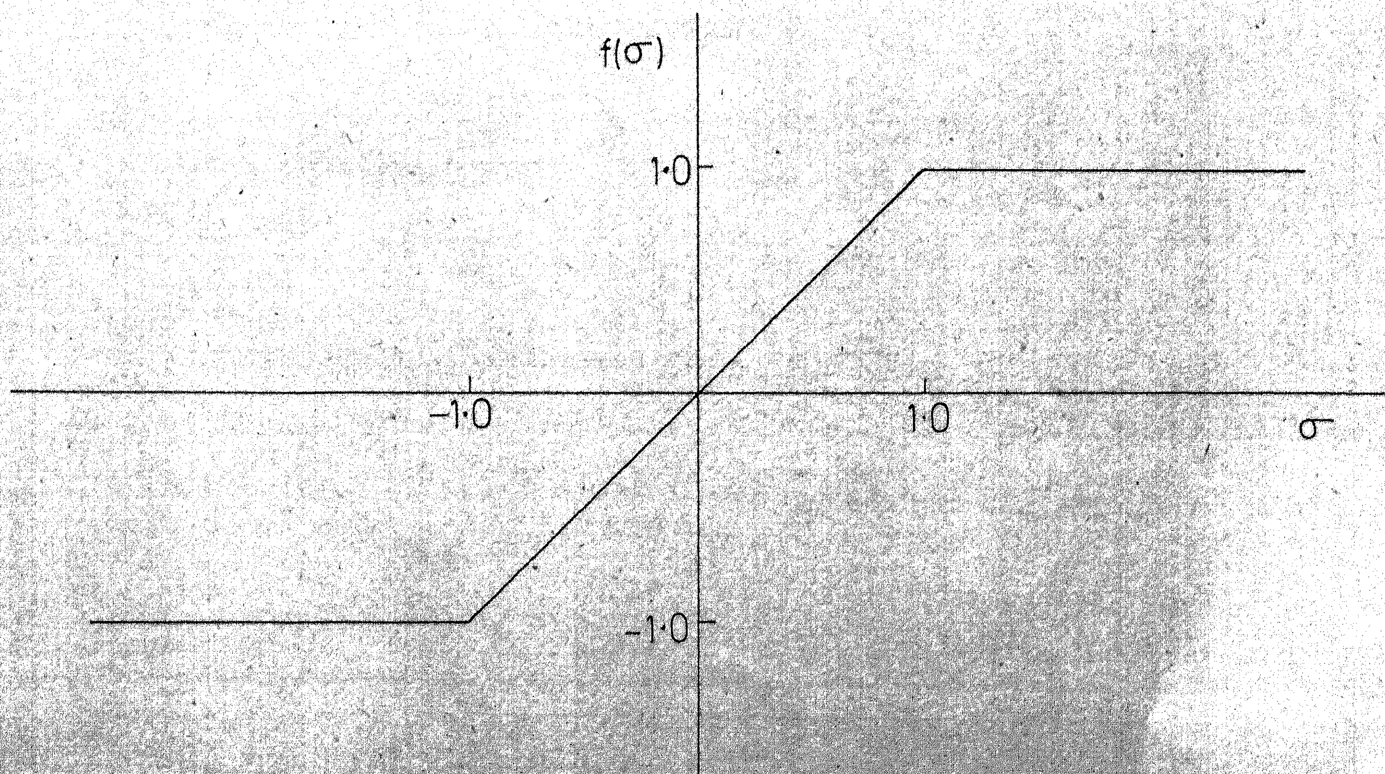


FIG.5.2 SATURATION OVERFLOW NONLINEARITY

As pointed out earlier, the interest is to study the nonlinear effects in a second order digital filter with saturation overflow nonlinearity with external forcing function for the coefficient values inside the stability triangle. This implies the transient response $x_t(k) \to 0$ as $k \to \infty$ irrespective of initial conditions and $x_s(k)$ will be the steady state solution of eqn. (5.2).

The transient term $x_t(k)$ in eqn. (5.3) is obtained as follows :

Let $z(k) = 0$, then the system equation (5.2) takes the form

$$x(k+2) - ax(k+1) - bx(k) = 0.$$

For this equation,

$$\lambda^2 - a\lambda - b = 0$$

is the characteristic equation, from which the roots are obtained as

$$\lambda_{1,2} = \frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 + b}.$$

The roots are oscillatory (complex conjucate) type if and only if b is negative and $|b| > (\frac{a}{2})^2$, that is

$$\lambda_{1,2} = \frac{a}{2} \pm j\sqrt{-b - (\frac{a}{2})^2}$$

from which

$$|\lambda| = r = \sqrt{-b}$$

and $\cos\beta = \frac{a}{2r}$, that is $a = 2r\cos\beta$.

where r and β are the amplitude and phase of the root.

Thus the transient solution is given by

$$x_t(k) = A\, r^k \cos\beta k + B r^k \sin\beta k \,. \qquad (5.4)$$

Here A and B are the constants to be evaluated from the initial data $x(o)$ and $x(1)$.

The steady state term $x_s(k)$ in (5.3) is evaluated as follows :

Let $z(k) = F\cos\alpha k$,

Assume the steady state solution of (5.2) as

$$x_s(k) = C\cos\alpha k + D\sin\alpha k \qquad (5.5)$$

where C and D are the constants that are related to the input variables namely, the amplitude F and the frequency α and their determination proceeds as follows :

From eqn. (5.5)

$$x_s(k+1) = (C\cos\alpha + D\sin\alpha)\cos\alpha k - (C\sin\alpha - D\cos\alpha)\sin\alpha k$$

$$x_s(k+2) = (C\cos 2\alpha + D\sin 2\alpha)\cos\alpha k - (C\sin 2\alpha - D\cos 2\alpha)\sin\alpha k \,,$$

Substituting the above expressions in eqn. (5.2) and collecting the like terms, the following equations are obtained.

$$C[\cos 2\alpha - a\cos\alpha - b] + D[\sin 2\alpha - a\sin\alpha] - F = 0.0$$

$$C[\sin 2\alpha - a\sin\alpha] - D[\cos 2\alpha - a\cos\alpha - b] = 0.0$$

from which

$$C = \frac{F(\cos 2\alpha - a\cos\alpha - b)}{1 + a^2 + b^2 - 2a(1-b)\cos\alpha - 2b\cos 2\alpha}$$

$$D = \frac{F(\sin 2\alpha - a\sin\alpha)}{1 + a^2 + b^2 - 2a(1-b)\cos\alpha - 2b\cos 2\alpha}.$$

Now the complete solution is

$$x(k) = Ar^k\cos\beta k + Br^k\sin\beta k + C\cos\alpha k + D\sin\alpha k.$$

The transient part of the above solution can be completely eliminated by proper choice of the initial conditions x(o) and x(1), since

$$x(o) = A + C$$

$$x(1) = Ar\cos\beta + Br\sin\beta + C\cos\alpha + D\sin\alpha$$

and by choosing

$$x(o) = C = F(\cos 2\alpha - a\cos\alpha - b)/R_1 \qquad (5.6)$$

$$x(1) = C\cos\alpha + D\sin\alpha = F[(1-b)\cos\alpha - a]/R_1 \qquad (5.7)$$

where

$$R_1 = 1 + a^2 + b^2 - 2a(1-b) \cos \alpha - 2b \cos 2\alpha. \qquad (5.8)$$

x(k) can now be rewritten as

$$x(k) = C \cos \alpha k + D \sin \alpha k$$

$$= R \cos (\alpha k + \theta)$$

where $$R^2 = \frac{F^2}{R_1} \qquad (5.9)$$

$$\theta = \tan^{-1} \left[\frac{\sin 2\alpha - a \sin \alpha}{\cos 2\alpha - a \cos \alpha - b}\right].$$

The following remark can now be made. In order to cause a jump from a linear oscillation (satisfying $f(\sigma) = \sigma$) to a nonlinear oscillation (where the saturation characteristic is effective) it is necessary that

$$R < 1.0 \qquad (5.10)$$

and R being real constrains $R_1 > 0$.

Further, if half a period of the input sinusoid is assumed to be an integer times the sample interval [95] then

$$\frac{\pi}{\alpha} = N \text{ (integer N)}.$$

Consider now the search for a particular type of nonlinear oscillation which has same frequency as input frequency and will remain clamped at the saturation level $\pm$ 1.0 at four

sampling instants per cycle (first two successive instants during positive half cycle and other two successive instants during negative half cycle). A mathematical statement of the prescribed nonlinear oscillation is

$$x(k) = 1.0 \qquad k = 0,1 \qquad (5.11)$$

$$x(k) = ax(k-1) + bx(k-2) + F \cos \alpha(k-2) \qquad k = 2,3, \ldots N-1 \qquad (5.12)$$

$$x(k) = -1.0 \qquad k = N,\ N+1 \qquad (5.13)$$

The above description essentially implies that the system is oscillating in the linear mode (i.e. $R < 1.0$) before disturbance application. The disturbance is injected at say $k = 0,1$ causing the response $x(k)$ to have unit value for $k = 0,1$.

It will now be shown how a particular choice of N enables determination in the a-b parameter plane, regions where transition from a linear solution with $R < 1.0$ to the above mentioned nonlinear oscillation take place. It is apparent that combination of conditions given in eqns. (5.11) to (5.13) together with the understanding that $R < 1.0$ from eqn. (5.10), should provide the conditions for jump to occur.

The method of deriving the required conditions is given below for various N.

Period 6 oscillation ($N = 3$) :

For $N = 3$, the following conditions are readily obtained

$$R_1 = 1 + a^2 + b^2 - a(1-b) + b > 0.0$$

$$x(k) = 1.0 \qquad k = 0,1$$

$$x(k) = ax(k-1) + bx(k-2) + F \cos \frac{\pi}{3} (k-2) \qquad k = 2$$

$$x(k) = -1.0 \qquad k = 3,4 \tag{5.14}$$

Thus

$$x(o) = 1.0$$

$$x(1) = 1.0$$

$$x(2) = a + b + F$$

$$x(3) = -1.0$$

$$x(4) = -1.0$$

Now from eqn. (5.1)

$$x(3) = f[ax(2) + bx(1) + F \cos \pi/3]$$

Thus using the earlier derived conditions

$$-1.0 = f[a(a+b+F) + b + F/2] \tag{5.15}$$

which implies

$$a(a+b+F) + b + F/2 < -1.0$$

from which

$$(a+0.5) F < - [a(a+b) + b + 1.0] . \tag{5.16}$$

Further

$$x(4) = f[ax(3) + bx(2) + F \cos 2\pi/3]$$

that is

$$-1.0 = f[-a + b(a+b+F) - F/2]$$

from which

$$-a + b(a+b+F) - F/2 < -1.0$$

$$(b - 0.5)F < -[b(a+b) - a + 1.0] . \qquad (5.17)$$

Thus we can state the necessary and sufficient conditions for a jump from a linear steady state oscillation with magnitude less than unity to a nonlinear oscillation (described above) as

$$C - 1 : \quad F^2 < R_1$$

$$C - 2 : \quad (a+0.5)F < -[a(a+b) + b + 1.0]$$

$$C - 3 : \quad (b-0.5)F < -[b(a+b) - a + 1.0] \qquad (5.18)$$

It is also obvious that satisfaction of conditions C-2, C-3 alone provides only the condition for sustained oscillations without necessarily the presence of a jump. Conditions C-1 provides the additional desired jump condition.

Period 8 oscillation (N = 4) :

With N = 4, the assumed relations in (5.11) - (5.13) assume the following form

$$x(k) = 1.0 \qquad k = 0,1$$
$$x(k) = ax(k-1) + bx(k-2) + F \cos\frac{\pi}{4}(k-2) \quad k = 2,3$$
$$x(k) = -1.0 \qquad k = 4,5 \tag{5.19}$$

Then

$$x(o) = 1.0$$
$$x(1) = 1.0$$
$$x(2) = a + b + F$$
$$x(3) = a(a+b+F) + b + F/\sqrt{2} = a(a+b) + b + (a+\frac{1}{\sqrt{2}})F$$
$$x(4) = -1.0$$
$$x(5) = -1.0$$

from the system equation (5.1)

$$x(4) = f[ax(3) + bx(2)]$$

from the above assumed relation $x(4) = -1.0$, we obtain

$$-1.0 = f[ax(3) + bx(2)]$$

that is

$$ax(3) + bx(2) < -1.0$$

which implies

$$a^2(a+b) + ab + a(a + \frac{1}{\sqrt{2}})F + b(a+b) + bF < -1.0$$
$$F(a^2 + \frac{a}{\sqrt{2}} + b) < -[a^2(a+b) + 2ab + b^2 + 1.0]. \tag{5.20}$$

Further

$$x(5) = -1.0 = f[ax(4) + bx(3) - F/\sqrt{2}]$$

that is

$$ax(4) + bx(3) - F/\sqrt{2} < -1.0$$

$$-a + ab(a+b) + b^2 + b(a + \frac{1}{\sqrt{2}})F - F/\sqrt{2} < -1.0$$

from which

$$F(ab + b/\sqrt{2} - \frac{1}{\sqrt{2}}) < -[ab(a+b) + b^2 - a + 1.0] \quad (5.21)$$

Then the necessary and sufficient condition for a jump to take place is

$$C - 1 \;:\; F^2 < R_1$$

$$C - 2 \;:\; F(a^2 + \frac{a}{\sqrt{2}} + b) < -[a^2(a+b) + 2ab + b^2 + 1.0]$$

$$C - 3 \;:\; F(ab + \frac{b}{\sqrt{2}} - \frac{1}{\sqrt{2}}) < -[ab(a+b) + b^2 - a + 1.0]$$

(5.22)

where

$$R_1 = 1 + a^2 + b^2 - \sqrt{2}\, a(1-b).$$

As in the case of period 6 oscillations, conditions C-1, C-2 and C-3 are the necessary and sufficient conditions for jump to take place and C-2 and C-3 are the necessary and sufficient conditions for the nonlinear oscillations to sustain.

Period 10 oscillation (N=5) :

Proceeding in the same way as above the following relations are obtained

$$x(k) = 1.0 \qquad k = 0,1$$
$$x(k) = ax(k-1) + bx(k-2) + F \cos \tfrac{\pi}{5}(k-2) \quad k = 2,3,4$$
$$x(k) = -1.0 \qquad k = 5,6 \tag{5.23}$$

from which

$$x(o) = 1.0$$
$$x(1) = 1.0$$
$$x(2) = a + b + F$$
$$x(3) = a(a+b) + b + F(a + 0.809)$$
$$x(4) = (a^2+b)(a+b) + ab + F(a^2 + 0.809a + b + 0.309)$$
$$x(5) = f[(a^3+2ab)(a+b) + a^2b + b^2 + F(a^3+0.809a^2 + 2ab + 0.309a + 0.809b - 0.309)]$$
$$x(6) = f[(a^2b+b^2)(a+b) +ab^2 - a + F(a^2b+0.809ab + b^2 + 0.309ab - 0.809)] .$$

Then imposing the conditions an $x(5)$ and $x(6)$ given in (5.23), we have

$$[(a^3+2ab)(a+b) + a^2b + b^2 + F(a^3 + 0.809a^2 + 2ab + 0.309a + 0.809b - 0.309)] < -1.0.$$

$$[(a^2b+b^2)(a+b) + ab^2 - a + F(a^2b + 0.809ab + b^2 + 0.309b - 0.809)] < -1.0 .$$

The necessary and sufficient conditions for jump phenomenon and nonlinear sustained oscillations can be obtained as follows from the above equations.

C-1 : $F^2 < R_1$

C-2 : $(a^3 + 0.809a^2 + 2ab + 0.309a + 0.809b - 0.309)F$

$$< -[1.0 + (a^3 + 2ab)(a+b) + a^2b + b^2]$$

C-3 : $(a^2b + 0.809ab + b^2 + 0.309b - 0.809)F$

$$< -[1.0 + (a^2b + b^2)(a+b) + ab^2 - a]$$

where

$$R_1 = 1 + a^2 + b^2 - 2a(1-b) \cos \pi/5 - 2b \cos 2\pi/5. \qquad (5.24)$$

Period 12 oscillations (N=6) :

The conditions stated in eqns. (5.11) - (5.13) take the following form for N=6.

$$x(k) = 1.0 \qquad k = 1,2$$
$$x(k) = ax(k-1) + bx(k-2) + F \cos \tfrac{\pi}{6}(k-2) \quad k = 2,3,4,5$$
$$x(k) = -1.0 \qquad k = 6,7 \qquad (5.25)$$

The above procedure can now be followed to obtain the conditions C-1, C-2 and C-3.

C-1 : $F^2 < R_1$

C-2 : $(a^4 + \frac{\sqrt{3}}{2} a^2 + 3a^2b + \frac{a^2}{2} + \sqrt{3}ab + b^2 + \frac{b}{2} - \frac{1}{2})F$

$$< -[1.0 + (a+b)(a^4+3a^2b+b^2) + ab(a^2+2b)]$$

C-3 : $(a^3b + \frac{\sqrt{3}}{2} a^2b + 2ab^2 + \frac{ab}{2} + \frac{\sqrt{3}}{2}b^2 - \frac{\sqrt{3}}{2})F$

$$< -[1.0+(a+b)(a^3b+2ab^2) + a^2b^2+b^3-a]$$

(5.26)

where

$$R_1 = 1 + a^2 + b^2 - \sqrt{3}a(1-b) - b.$$

Here again all the above three conditions are necessary and sufficient to cause a jump and the last two conditions are necessary in order to cause nonlinear sustained oscillations.

The conditions stated in eqns. (5.11) - (5.13) are the most general for all N. For N = 3,4,5 and 6, the eqns. (5.18), (5.22) (5.24) and (5.26) respectively provide the necessary conditions. From these conditions the following useful results can be deduced.

(i) For given input variables (F and α), the values of filter coefficients [a,b] satisfying all the three inequalities C-1 to C-3, give a region in the a-b parameter plane in which a jump behaviour is observed.

(ii) For given input frequency and amplitude, the satisfaction of conditions C-2 and C-3 alone gives a region where there exists a nonlinear sustained oscillation.

(iii) For a given input frequency and given [a,b] values, the values of F, if any, satisfying all the three inequalities C-1 to C-3 provide the range of F for which there exists always a jump in the solution.

The conditions C-1 to C-3 (assuming equality in their respective definitions) are plotted in the a-b plane for

various values of the input amplitude F. Fig. 5.3 gives the complete plot of all the three conditions stated in (5.18) for N = 3, whereas Figs. 5.4, 5.5 and 5.6 provide the plot of all the conditions given in eqns. (5.22), (5.24) and (5.26), all for a particular F. The complete computed values of eqns. (5.22), (5.24) and (5.26) are tabulated in Tables 5.1, 5.2 and 5.3 respectively.

For illustration, for N = 3 and for F = -0.5, the region ABCDE in Fig. 5.3 provides a region where the nonlinear phenomena exist. The region CDEC gives the region where the jump phenomenon is possible and the region ABCEA gives the region for nonlinear sustained oscillation. For other values of N, the above regions are indicated in the respective figures. The line PQ in Fig. 5.3 and the curve PQ in Fig. 5.4 define the locus of the tip of the boundary of the region where the nonlinear sustained oscillation is possible.

Once the values of input amplitude and the input frequency are known, the region in which the jump phenomenon takes place can be determined as mentioned in the previous section. Here the jump behaviour is observed by applying the perturbation in the system variable such that $x(j) = 1.0$ for $j = 2Nn$, $2Nn + 1$ for a particular integer $n \geq 1$. It is to be noted that the conditions C-1 to C-3 are derived based on the assumption that $x(o) = x(1) = 1.0$ when the nonlinearity is effective

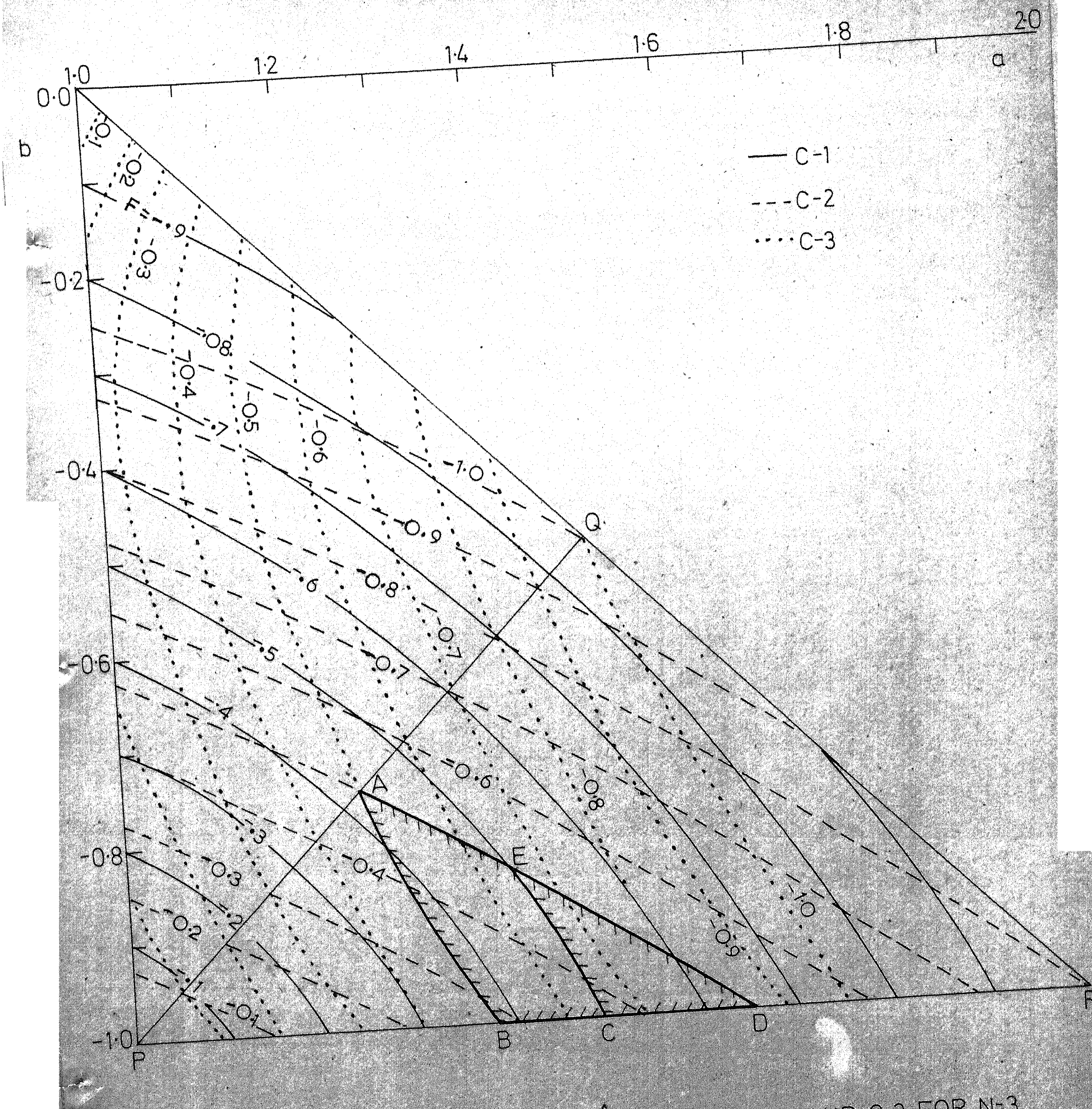


FIG.5·3 LOCUS OF THE CONDITIONS C-1, C-2 AND C-3 FOR N=3

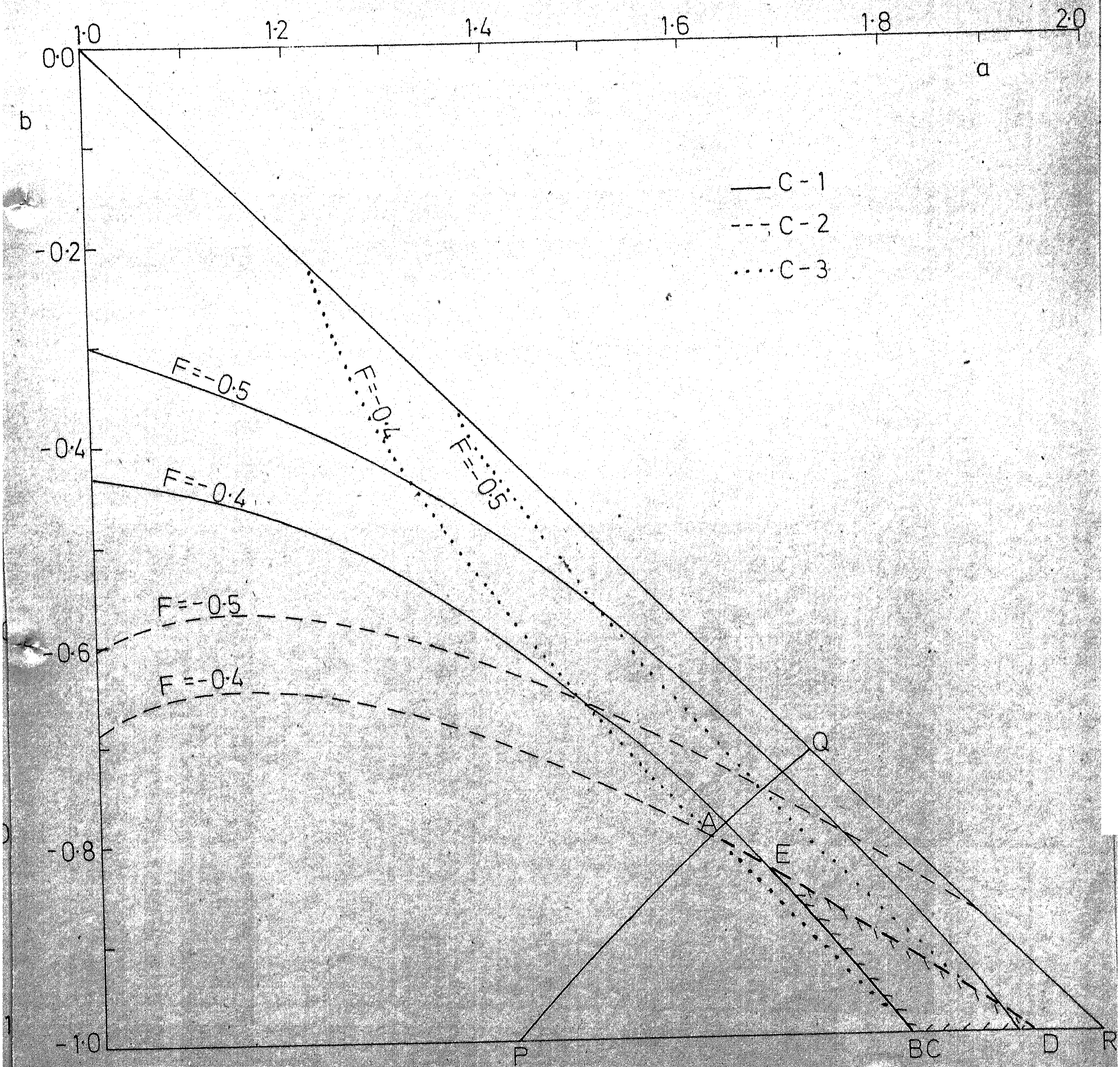


FIG.5.4 LOCUS OF CONDITIONS C-1, C-2 AND C-3 FOR N=4

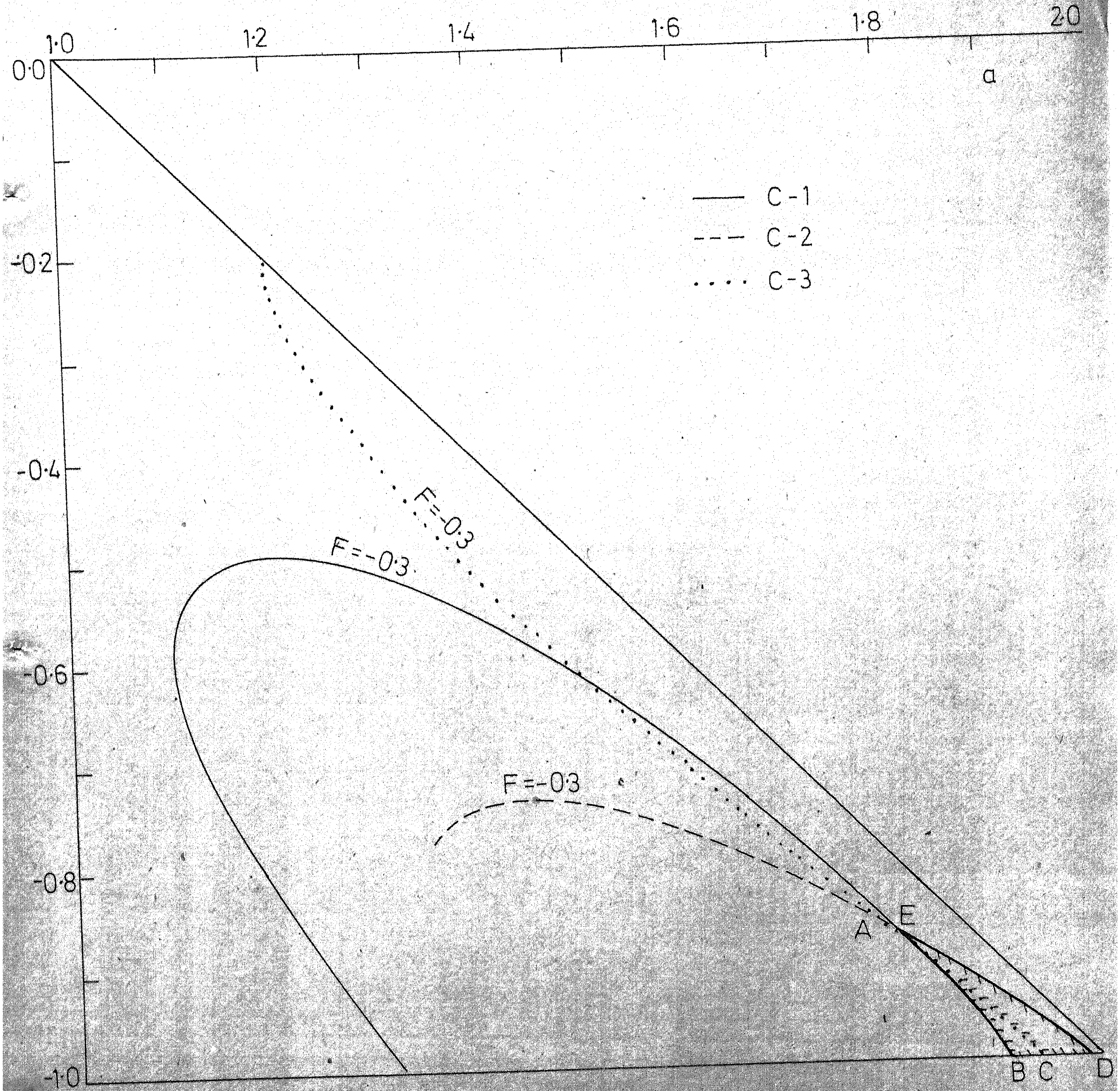


FIG. 5·5 LOCUS OF CONDITIONS C-1, C-2 AND C-3 FOR N=5

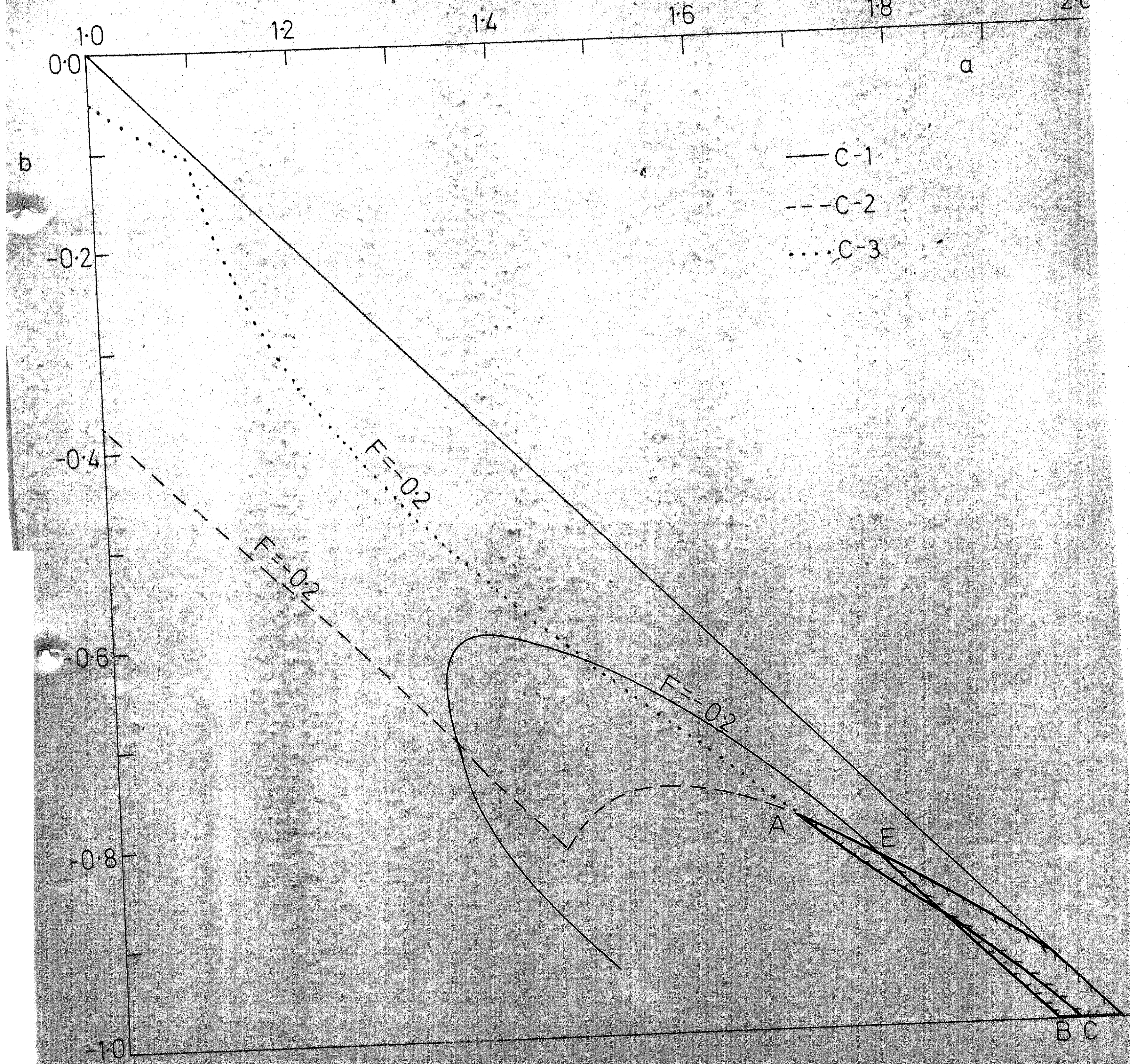


FIG.5·6 LOCUS OF CONDITIONS C-1, C-2 AND C-3 FOR N=6

Table 5.1a COMPUTED VALVES FOR CONDITION C-1, N = 4

a	VALUE OF b						
	F=-0.7	F=-0.6	F=-0.5	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00	-0.0713	-0.1835	-0.3019	-0.4347 -0.9795	-0.6424		
1.05	-0.0916	-0.2006	-0.3139	-0.4364	-0.5867 -0.8962		
1.10	-0.1140	-0.2205	-0.3299	-0.4452	-0.5763 -0.9793		
1.15		-0.2430	-0.3494	-0.4595	-0.5785	-0.7420 -0.8844	
1.20		-0.2680	-0.3721	-0.4784	-0.5896	-0.7180 -0.9790	
1.25		-0.2953	-0.3976	-0.5011	-0.6073	-0.7211	
1.30		-0.3247	-0.4258	-0.5225	-0.6304	-0.7365	-0.8606 -0.9779
1.35		-0.3563	-0.4567	-0.5572	-0.6581	-0.7600	-0.8657
1.40			-0.4901	-0.5901	-0.6902	-0.7903	-0.8906
1.45			-0.5260	-2.6262	-0.7264	-0.8270	-0.9288
1.50			-0.5644	-0.6653	-0.7669	-0.8702	-0.9814
1.55			-0.6054	-0.7078	-0.8119	-0.9207	

a	VALUES OF b						
	F=-0.7	F=-0.6	F=-0.5	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.60			-0.6490	-0.7536	-0.8617	-0.9807	
1.65			-0.6954	-0.8032	-0.9174		
1.70			-0.7448	-0.8569	-0.9803		
1.75			-0.7975	-0.9156			
1.80			-0.8538	-0.9803			
1.85			-0.9145				
1.90			-0.9803				
1.95							
2.00							

Table 5.1b COMPUTED VALUES FOR CONDITION C-2, N = 4

a	VALUES OF b						
	F=-0.7	F=-0.6	F=-0.5	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00	-0.4307	-0.5187	-0.6050	-0.6894	-0.7716	-0.8510	-0.9273
1.05	-0.4150	-0.4997	-0.5827	-0.6638	-0.7428	-0.8193	-0.8930
1.10	-0.4082	-0.4909	-0.5719	-0.6511	-0.7284	-0.8039	-0.8760
1.15	-0.4080	-0.4893	-0.5691	-0.6472	-0.7234	-0.7977	-0.8698
1.20	-0.4129	-0.4933	-0.5722	-0.6496	-0.7253	-0.7992	-0.8711
1.25	-0.4219	-0.5017	-0.5801	-0.6570	-0.7324	-0.8062	-0.8781
1.30	-0.4393	-0.5137	-0.5917	-0.6684	-0.7437	-0.8175	-0.8896
1.35	-0.4496	-0.5287	-0.6065	-0.6832	-0.7585	-0.8323	-0.9048
1.40	-0.4674	-0.5463	-0.6241	-0.7007	-0.7761	-0.8502	-0.9229
1.45	-0.4873	-0.5662	-0.6440	-0.7207	-0.7962	-0.8706	-0.9437
1.50	-0.5092	-0.5881	-0.6660	-0.7428	-0.8186	-0.8932	-0.9667
1.55		-0.6118	-0.6898	-0.7668	-0.8928	-0.9178	-0.9917
1.60		-0.6370	-0.7152	-0.7924	-0.8688	-0.9441	

a	VALUES OF b						
	F=−0.7	F=−0.6	F=−0.5	F=−0.4	F=−0.3	F=−0.2	F=−0.1
1.65		−0.6638	−0.7421	−0.8196	−0.8962	−0.9719	
1.70			−0.7704	−0.8482	−0.9251		
1.75			−0.8000	−0.8780	−0.9552		
1.80			−0.8305	−0.9088	−0.9864		
1.85			−0.8622	−0.9408			
1.90				−0.9737			
1.95							
2.00							

Table 5.1c COMPUTED VALUES FOR CONDITION C-3, N=4

a	VALUES of b					
	F=-0.55	F=-0.5	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00						-0.1200 -0.2947
1.05						-0.4285
1.10					-0.3473	-0.5171
1.15				-0.2309	-0.4512	-0.5911
1.20				-0.3800	-0.5313	-0.6573
1.25			-0.2975	-0.4694	-0.6008	-0.7184
1.30			-0.4042	-0.5433	-0.6641	-0.7760
1.35			-0.4844	-0.6093	-0.7232	-0.8310
1.40		-0.4235	-0.5537	-0.6701	-0.7793	-0.8841
1.45		-0.4972	-0.6167	-0.7275	-0.8333	-0.9356
1.50	-0.5029	-0.5627	-0.6755	-0.7884	-0.8654	-0.9859
1.55	-0.5668	-0.6233	-0.7314	-0.8353	-0.9323	
1.60	-0.6263	-0.6803	-0.7851	-0.8867	-0.9860	
1.65	-0.6826	-0.7349	-0.8371	-0.9369		
1.70	-0.7365	-0.7876	-0.8878	-0.9862		
1.75	-0.7887	-0.8388	-0.9375			
1.80	-0.8396	-0.8889	-0.9863			
1.85	-0.8894	-0.9380				
1.90	-0.9383	-0.9864				
1.95	-0.9864					
2.00						

Table 5.2a SIMULATED VALUES FOR CONDITION C-1, N = 5

a	VALUES OF b				
	F=-0.5	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00	-0.8[illegible]36 -0.1564	-0.6674 -0.3326			
1.05	-0.9126 -0.1683	-0.7607 -0.3202			
1.10	-0.9775 -0.1843	-0.8403 -0.3215	-0.5809		
1.15	-0.2038	-0.9117 -0.3310	-0.7410 -0.5017		
1.20	-0.2263	-0.9774 -0.3462	-0.8339 -0.4897		
1.25	-0.2515	-0.3658	-0.9101 -0.4944	-0.7023	
1.30		-0.4155	-0.5279	-0.9064 -0.6600	
1.40		-0.4447	-0.5524	-0.9772 -0.6701	-0.8236
1.45		-0.4764	-0.5808	-0.6902	-0.8484 -0.8797
1.50		-0.5106	-0.6126	-0.7169	-0.8325 -0.9765
1.55			-0.6476	-0.7490	-0.8533
1.60			-0.6856	-0.7857	-0.8860
1.65			-0.7265	-0.8268	-0.9276
1.70			-0.7702	-0.8722	-0.97867
1.75			-0.8170	-0.9224	
1.80			-0.8669	-0.9782	
1.85			-0.9204		
1.90			-0.9781		
1.95					
2.00					

Table 5.2b SIMULATED VALUES OF CONDITION C-2, N = 5

a	VALUES OF b						
	F=-0.7	F=-0.6	F=-0.5	F=-0.4	F=-0.3	F=0.2	F=0.1
1.00							
1.05							
1.10							
1.15							
1.20	-0.4291 -0.6891						
1.25	-0.3785 -0.8570	-0.5109 -0.8192					
1.30	-0.3625 -0.9932	-0.4717 -0.9788	-0.5892 -0.9560	-0.7336 -0.9062			
1.35	-0.3604	-0.4601	-0.5616	-0.6662	-0.7774	-0.9124	
1.40		-0.4609	-0.5553	-0.6494	-0.7431	-0.8363	-0.9285
1.45		-0.4693	-0.5595	-0.6484	-0.7358	-0.8206	-0.9020
1.50			-0.5704	-0.6564	-0.7404	-0.8216	-0.8992
1.55			-0.5862	-0.6703	-0.7522	-0.8315	-0.9073
1.60			-0.6056	-0.6885	-0.7692	-0.8473	-0.9222
1.65				-0.7100	-0.7899	-0.8674	-0.9420
1.70				-0.7343	-0.8137	-0.8909	-0.9653
1.75				-0.7608	-0.8400	-0.9170	-0.9915
1.80					-0.8684	-0.9454	
1.85					-0.8985	-0.9756	
1.90					-0.9302		
1.95					-0.9633		
2.00							

Table 5.2c SIMULATED VALUES FOR THE CONDITION C-3, N = 5

a	VALUES OF b			
	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00			-0.0988	-0.1451
1.05			-0.1204	-0.2943
1.10			-0.1415	-0.3901
1.15			-0.3121	-0.4621
1.20		-0.2144	-0.3947	-0.5234
1.25		-0.3230	-0.4609	-0.5784
1.30		-0.3969	-0.5189	-0.6291
1.35		-0.4589	-0.5718	-0.6769
1.40		-0.5143	-0.6213	-0.7227
1.45	-0.4563	-0.5656	-0.6683	-0.7669
1.50	-0.5097	-0.6140	-0.7136	-0.8099
1.55	-0.5596	-0.6604	-0.7576	-0.8522
1.60	-0.6072	-0.7053	-0.8007	-0.8939
1.65	-0.6531	-0.7492	-0.8431	-0.9352
1.70		-0.7923	-0.8850	-0.9762
1.75		-0.8348	-0.9266	
1.80		-0.8769	-0.9680	
1.85		-0.9188		
1.90		-0.9605		
1.95				
2.00				

Table 5.3a SIMULATED VALUES FOR CONDITION C-1, N = 6

a	VALUES OF b				
	F=-0.5	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00	-0.0254 -0.7066	-0.2047 -0.5273			
1.05	-0.7750	-0.2003 -0.6184			
1.10	-0.8401	-0.2074 -0.6978			
1.15	-0.9025	-0.0215 -0.7709	-0.4231 -0.5688		
1.20	-0.9626	-0.2405 -0.8379	-0.4006 -0.6779		
1.25		-0.2633 -0.9018	-0.4039 -0.7611		
1.30		-0.9625	-0.4177 -0.8340		
1.35			-0.4378 -0.9004	-0.6099 -0.7284	
1.40			-0.4626 -0.9623	-0.6009 -0.8239	
1.45			-0.4909	-0.6139 -0.8975	
1.50			-0.5224	-0.6361 -0.9619	
1.55			-0.5565	-0.6643	-0.8009 -0.8837
1.60				-0.6968	-0.8105 -0.9607
1.65				-0.7332	-0.8377
1.70				-0.7129	-0.8735
1.75				-0.8157	-0.9159
1.80				-0.8617	-0.9648
1.85				-0.9110	
1.90				-0.9639	
1.95					
2.00					

Table 5.3b SIMULATED VALUES FOR CONDITION C-2, N = 6

	VALUES OF b			
	F=-0.4	F=-0.3	F=-0.2	F=-0.1
1.00	-0.3000	-0.3388	-0.3770	-0.4147
1.05	-0.3451	-0.3836	-0.4217	-0.4595
1.10	-0.3905	-0.4290	-0.4672	-0.5050
1.15	-0.4368	-0.4753	-0.5135	-0.5514
1.20	-0.4839	-0.5224	-0.5607	-0.5987
1.25	-0.5319	-0.5705	-0.6089	-0.6470
1.30	-0.5808	-0.6195	-0.6580	-0.6963
1.35	-0.6306	-0.6695	-0.7081	-0.7465
1.40	-0.5412	-0.7205	-0.7592	-0.7978
1.45	-0.5302	-0.6561	-0.8113	-0.8500
1.50		-0.6385	-0.7641	-0.9033
1.55		-0.6444	-0.7498	-0.8870
1.60		-0.6590	-0.7564	-0.8650
1.65		-0.6791	-0.7721	-0.8713
1.70		-0.7030	-0.7936	-0.8880
1.75			-0.8190	-0.9106
1.80			-0.8474	-0.9373
1.85			-0.8782	-0.9671
1.90			-0.9110	-0.9992
1.95			-0	
2.00				

Table 5.3c SIMULATED VALUES FOR CONDITION C-3, N = 6

a	VALUES OF b		
	F=-0.3	F=-0.2	F=-0.1
1.00		-0.0582	-0.1113
1.05		-0.0859	-0.2290
1.10		-0.1121	-0.3392
1.15		-0.2501	-0.4149
1.20		-0.3389	-0.4764
1.25	-0.25914	-0.4056	-0.5296
1.30	-0.3352	-0.4619	-0.5771
1.35	-0.3959	-0.5120	-0.6208
1.40	-0.4488	-0.5580	-0.6620
1.45	-0.4969	-0.6011	-0.7013
1.50	-0.5419	-0.6423	-0.7395
1.55	-0.5847	-0.6823	-0.7770
1.60	-0.6261	-0.7214	-0.8142
1.65	-0.6666	-0.7600	-0.8513
1.70	-0.7064	-0.7984	-0.8885
1.75		-0.8368	-0.9259
1.80		-0.8752	-0.9636
1.85		-0.9139	
1.90		-0.952[illegible]	
1.95		-0.9921	
2.00			

112]. Depending on the time instants at which $x(j) = 1.0$ and $x(j+1) = 1.0$ are applied for $j = 2Nn$, sometimes transients may be observed when the response jumps from one steady state solution to another. In the present study the magnitude of the disturbance to be applied may be large and it is of interest to ascertain how small a disturbance can cause the above prescribed nonlinear oscillation.

Minimum Disturbance required to cause nonlinear oscillations :

Instead of disturbing the system response at time instants j, j+1 such that $x(j) = 1.0$ and $x(j+1) = 1.0$, one can try to obtain the smallest values of the disturbances for which the same response as above can be obtained under steady state. The procedure to obtain these conditions are as follows :

Consider the given system and let the initial values be $x(o) = x_o$ and $x(1) = x_1$, which are to be determined such that the output sequence assumes the form given in eqns. (5.11) to (5.13). Then,

$$
\begin{aligned}
x(2) &= ax_1 + bx_o + F \\
x(3) &= ax(2) + bx_1 + F \cos \alpha \\
x(4) &= ax(3) + bx(2) + F \cos 2\alpha \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \qquad (5.27)
\end{aligned}
$$

Eqn. (5.27) is a set of expressions for which $|x(j)|$ is constrained to be less than or equal to unity. That is although each of the equations in (5.27) describes a linear system, a properly combined version may lead to a nonlinear oscillation. Then knowing the region in the a-b plane for a specified input amplitude and frequency, the required minimum values for x_o and x_1 can be calculated using the relations given in (5.27). For example.

(i) Let N = 3, the period 6 oscillation :

The expressions in the eqn. (5.27) take the following form

$$x(2) = ax_1 + bx_o + F \tag{5.28}$$

$$x(3) = -1.0 = ax(2) + bx_1 + F\cos\pi/3 = ax(2) + bx_1 + F/ \tag{5.29}$$

$$x(4) = -1.0 = ax(3) + bx(2) - F/2 = -a + bx(2) - F/2 \tag{5.30}$$

from the last expression

$$x(2) = (a + F/2 - 1.0)/b$$

and from equations (5.29) and (5.28)

$$x_1 = -(1.0 + ax(2) + F/2)/b \tag{5.31}$$

$$x_o = (x(2) - ax_1 - F)/b \tag{5.32}$$

The eqns. (5.31) and (5.32) provide the minimum values to

and for $N = 6$

$$x(5) = (a + \frac{\sqrt{3}}{2} F - 1.0)/b$$

$$x(4) = -(1.0 + ax(5) - F/2)/b$$

$$x(3) = (x(5) - ax(4)/b$$

$$x(2) = (x(4) - ax(3) - F/2)/b$$

$$x_1 = (x(3) - ax(2) - \frac{\sqrt{3}}{2} F)/b \tag{5.38}$$

$$x_o = (x(2) - ax_1 - F)/b. \tag{5.39}$$

Then with these x_o and x_1 values and for the known filter coefficients [a,b] in the region in which the jump or non-linear sustained oscillations are possible, the time response of the system equation (5.1) is computed and the results are discussed in the following section.

5.5 Simulation Results :

The theoretical investigations reported in the above section are verified by actually simulating the system equation as a second order recurrence relation. The following are the results corresponding to the observations made in the previous section.

5.5.1 Jump up phenomena :

The filter parameters [a,b] are chosen from the region where the jumps are possible for specified input amplitude, and the minimum values of x_o and x_1 required to cause the

nonlinear oscillations are obtained for various N as indicated in section 5.4 and using these values such that $x(j) = x_o$ and $x(j+1) = x_1$, the time response of the given system is simulated for various discrete time index k. For all these cases the linear response, whose steady state amplitude is less than unity is evaluated using the initial conditions x(o) and x(1) given in eqns. (5.6) and (5.7).

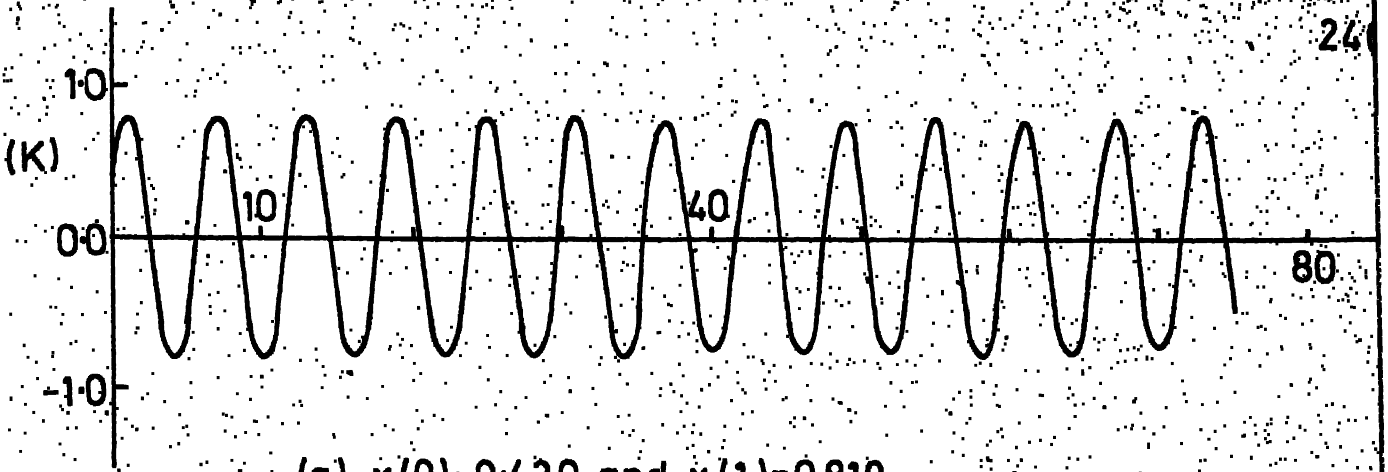
Fig. 5.3 shows a region CDEC obtained in the a-b parameter plane, the interior of which satisfies inequalities C-1 to C-3 for N = 3. This above region is obtained for F = -0.5. Then consider a = 1.6 and b = -0.9 inside CDEC, with these values and from eqns. (5.31) and (5.32), we have

$$x_o = 0.152$$
$$x_1 = 0.182 \qquad (5.40)$$

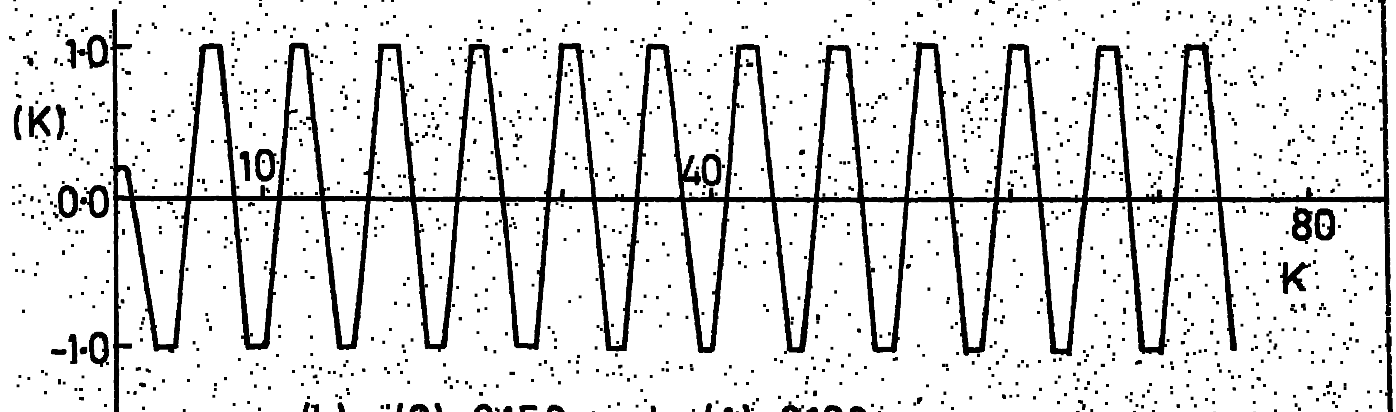
and from eqns. (5.6) and (5.7) the following initial values are obtained

$$x(o) = 0.43$$
$$x(1) = 0.819 \qquad (5.41)$$

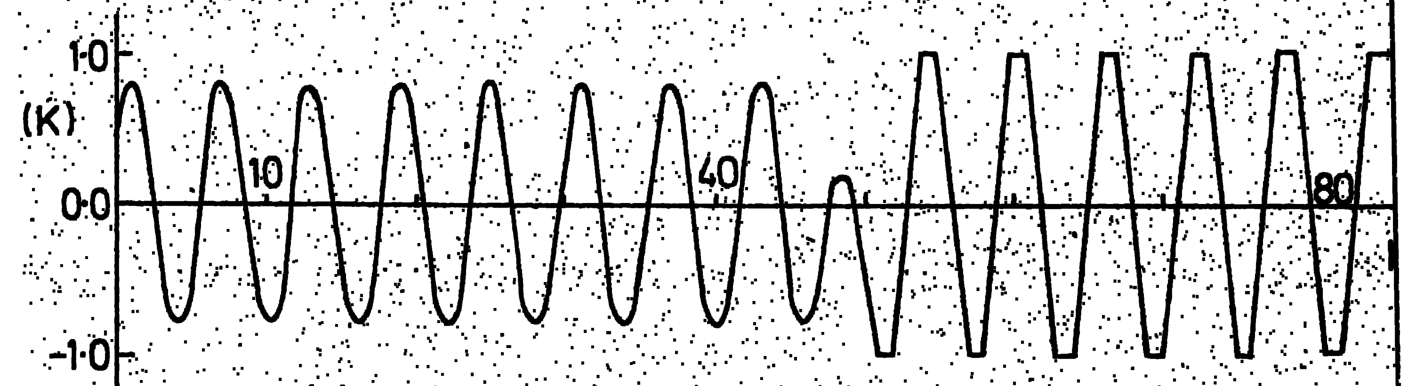
Fig. 5.7a gives the time response with initial conditions given in (5.41) and Fig. 5.7b shows the time plot with $x(o) = x_o$ and $x(1) = x_1$, where x_o and x_1 are given in eqn. (5.40). Figs. 5.7c to 5.7h show a jump up phenomena between the solutions when the x_o and x_1 values are applied such that $x(j) = x_o$ and



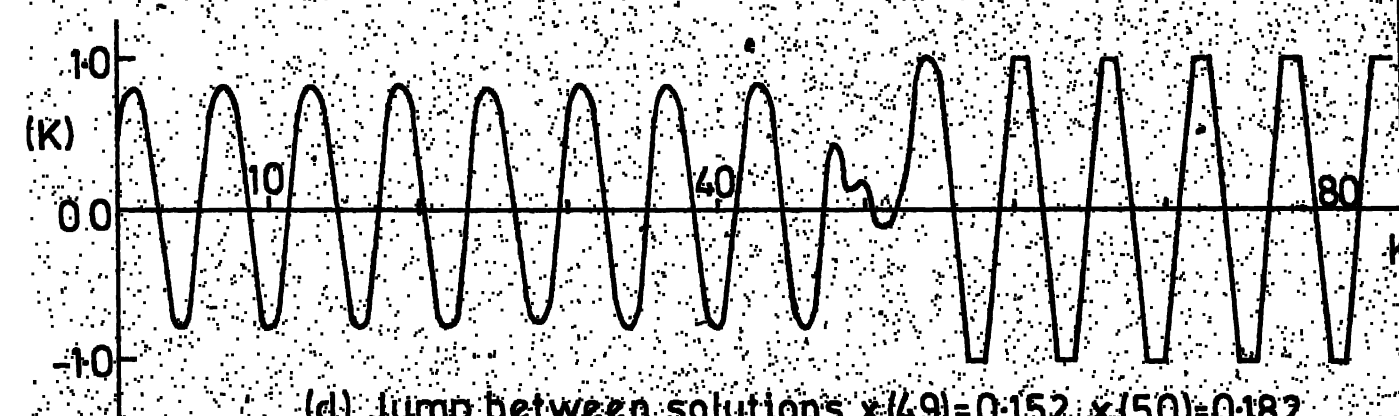
(a) x(0)=0·430 and x(1)=0·819



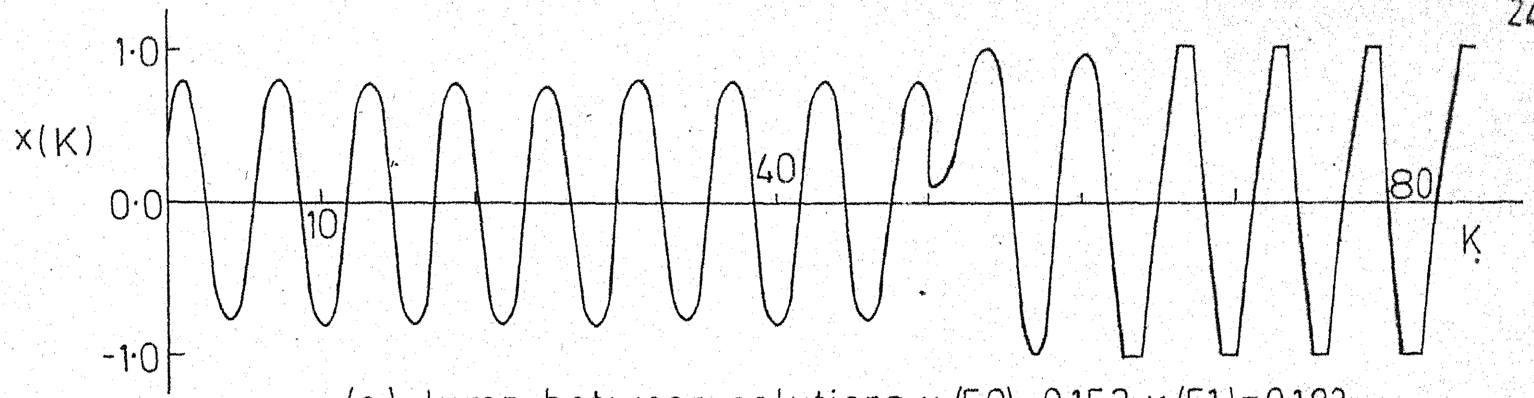
(b) x(0)=0·152 and x(1)=0·182



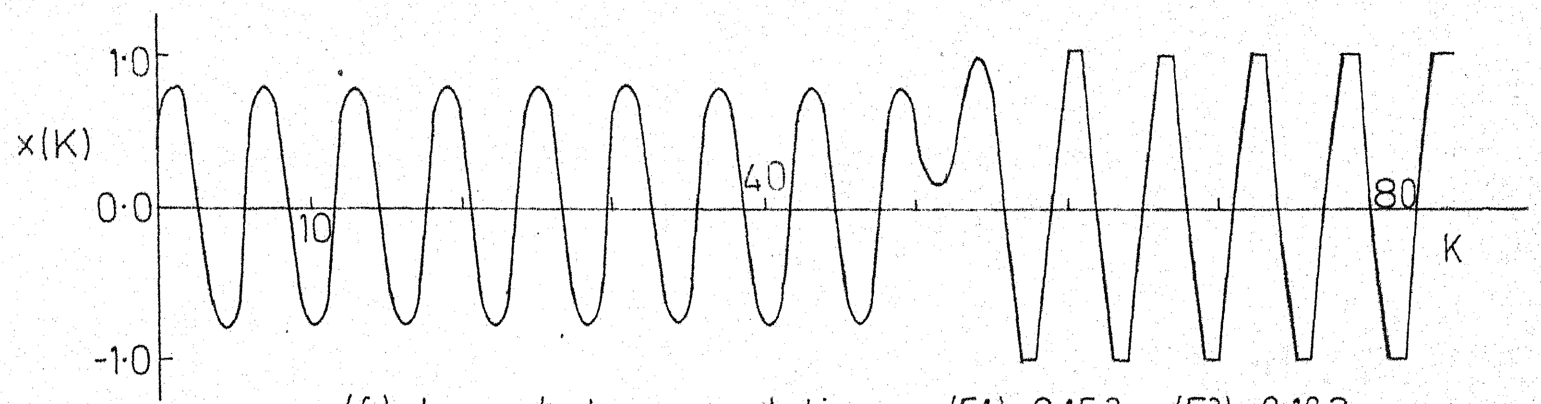
(c) Jump between solutions x(48)=0·152, x(49)=0·182



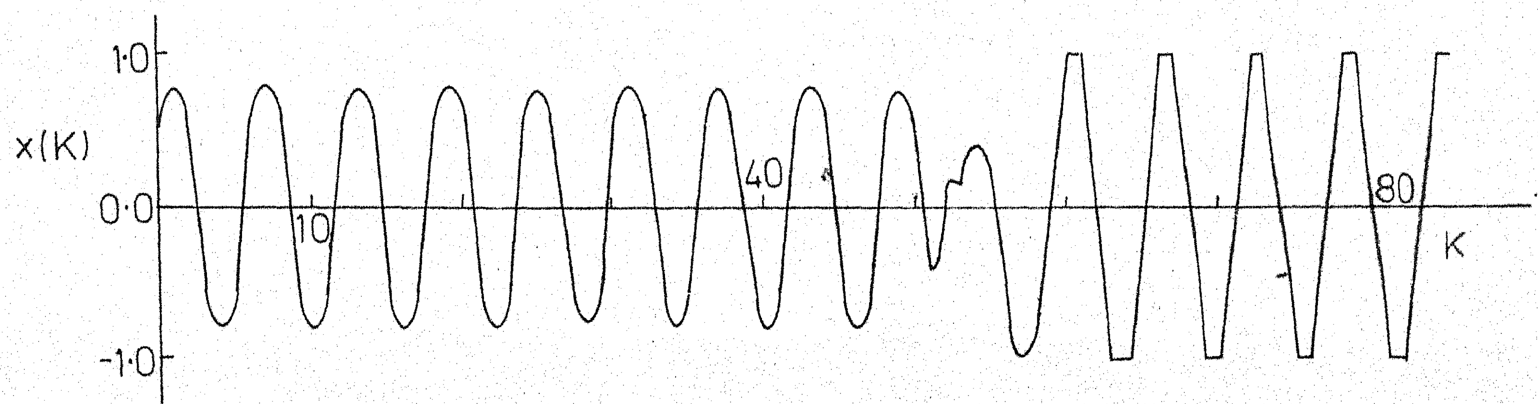
(d) Jump between solutions x(49)=0·152, x(50)=0·182



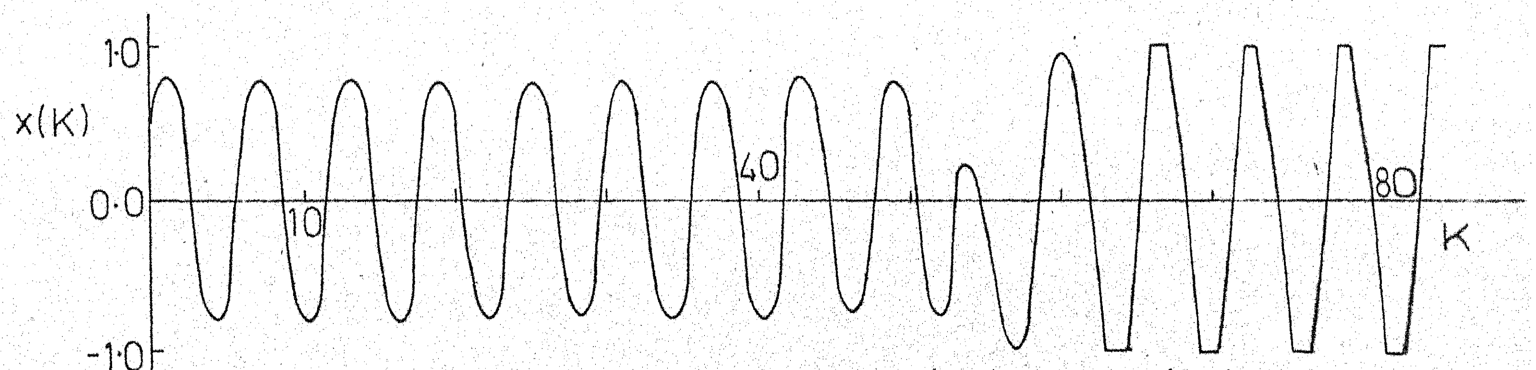
(e) Jump between solutions x (50) = 0·152, x (51) = 0·182



(f) Jump between solutions x (51) = 0·152, x (52) = 0·182



(g) Jump between solutions x (52) = 0·152, x (53) = 0·182



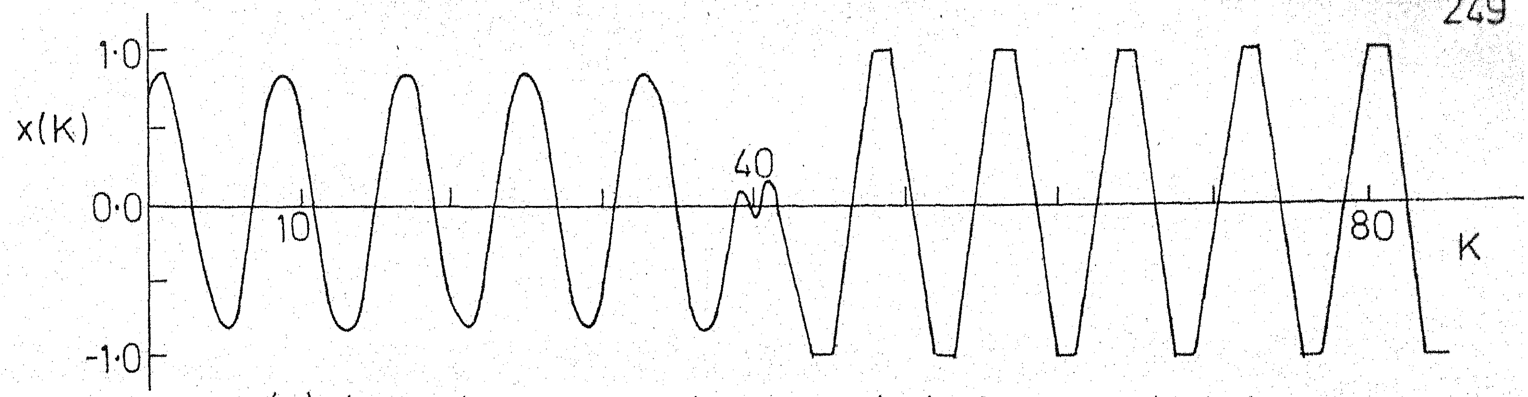
(h) Jump between solutions x (53) = 0·152, x (54) = 0·182

FIG. 5·7 SOLUTION TO EQN. (51) WITH a = 1·6, b = 0·98, F = −0·5

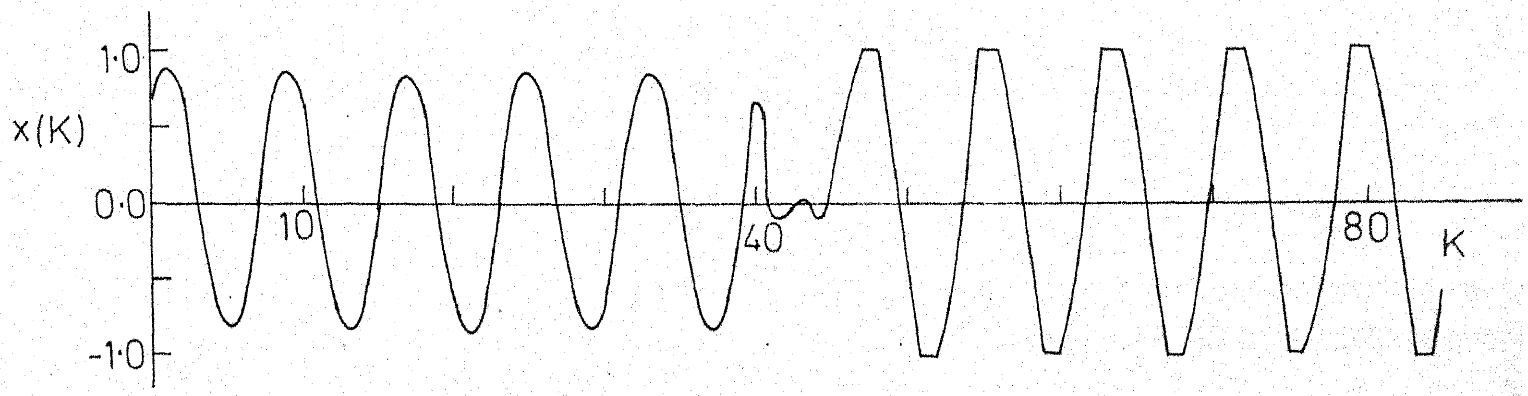
$x(j+1) = x_1$ where j is the integer valued time instant. It is observed by simulation that for $j = 2\,nN$ $(n = 1,2, \ldots)$ a jump up phenomenon is observed and the steady state nonlinear oscillations has the description given earlier after a few cycle of transient behaviour. The Figs. 5.7c to 5.7h show the jump behaviour between the steady state solutions when x_o and x_1 are applied at various time instants such that $x(j) = x_o$ and $x(j+1) = x_1$ for $j = 2nN + m$, that is for $n = 6$ and $m = 0,1,2, \ldots 5$ respectively. It is interesting to observe the transient behaviour that is developed before the steady state nonlinear oscillations sets in.

For $N = 4$, similar results are plotted in Figs. 5.8a to 5.8h with $a = 1.85$, $b = -0.95$, $x(o) = 0.606$, $x(1) = 0.852$, $x_o = -0.0784$, $x_1 = 0.1165$ and for $F = -0.4$ with the disturbance application $x(j) = x_o$, $x(j+1) = x_1$ where $j = 2nN+m$, $n = 5$, $m = 0,1,2, \ldots 7$ respectively. Here again the transients between two steady state solutions are observed when the disturbance is applied at various discrete time instants.

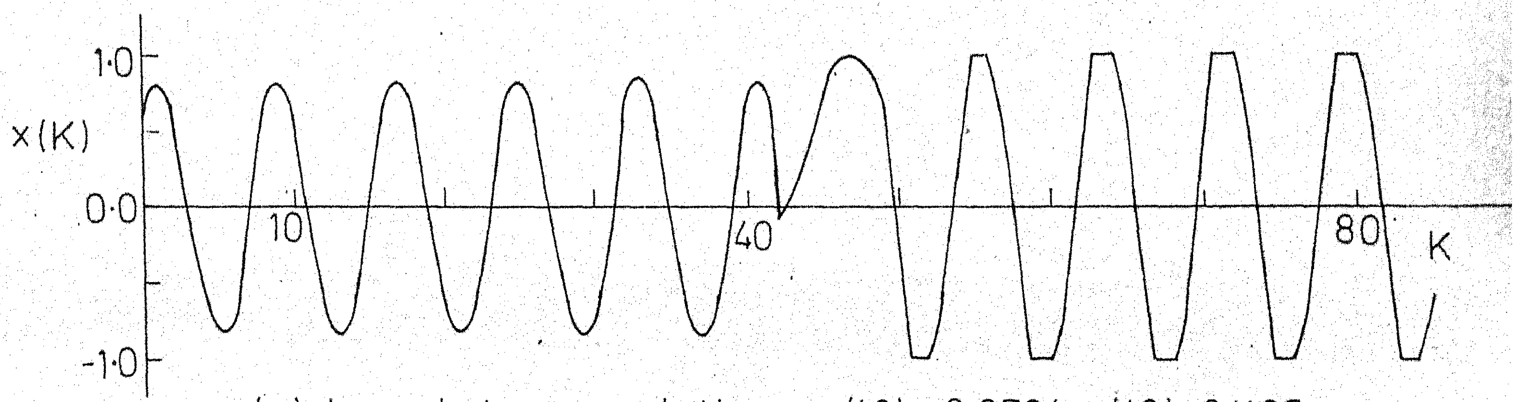
For $N = 5,6$, the results obtained are verified with exact simulation and the time plots showing the jump behaviour are not given here.



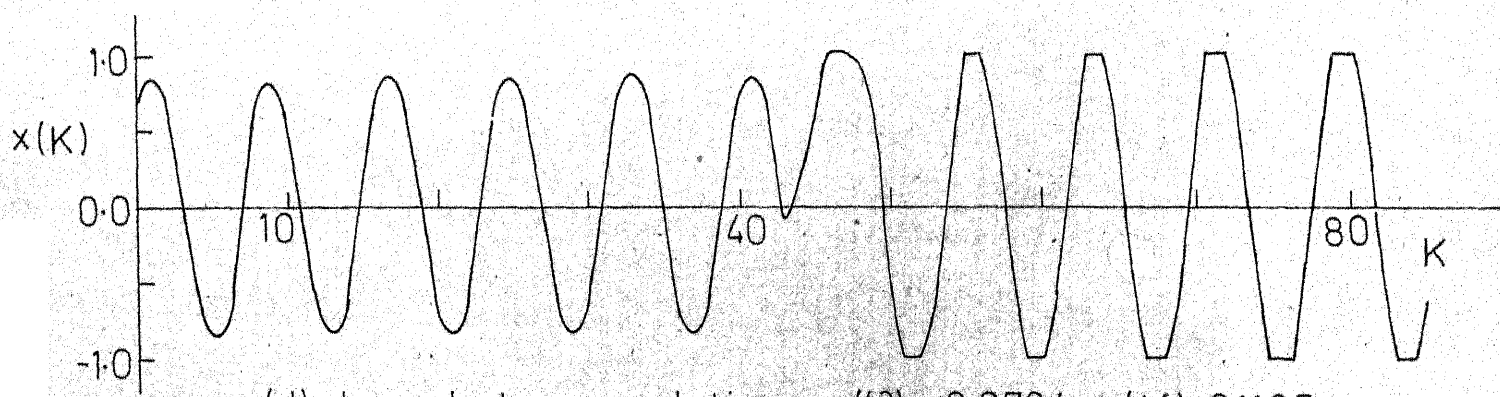
(a) Jump between solutions x(40)=-0·0784, x(41)=0·1165



(b) Jump between solutions x(41)=-0·0784, x(42)=0·1165

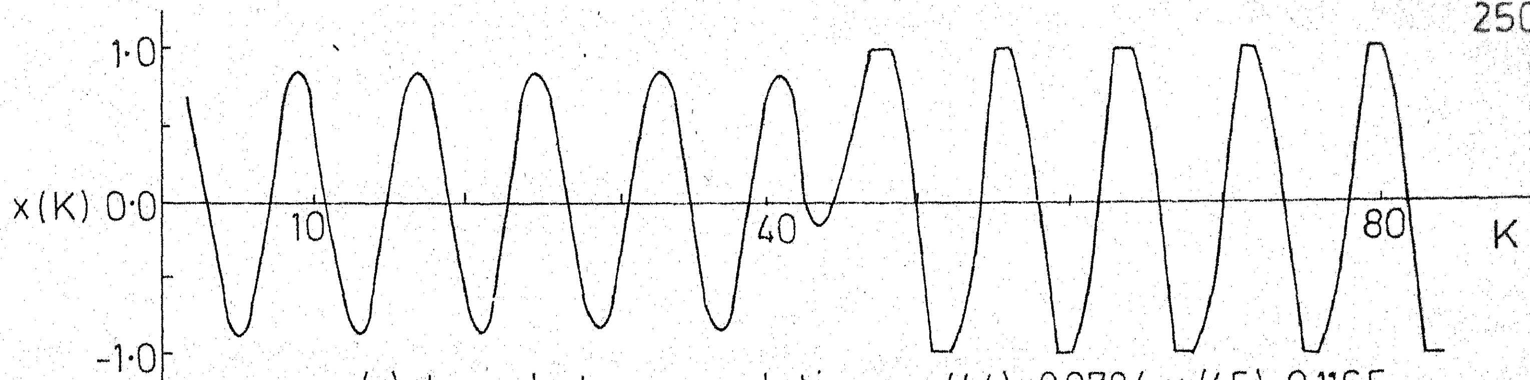


(c) Jump between solutions x(42)=-0·0784, x(43)=0·1165

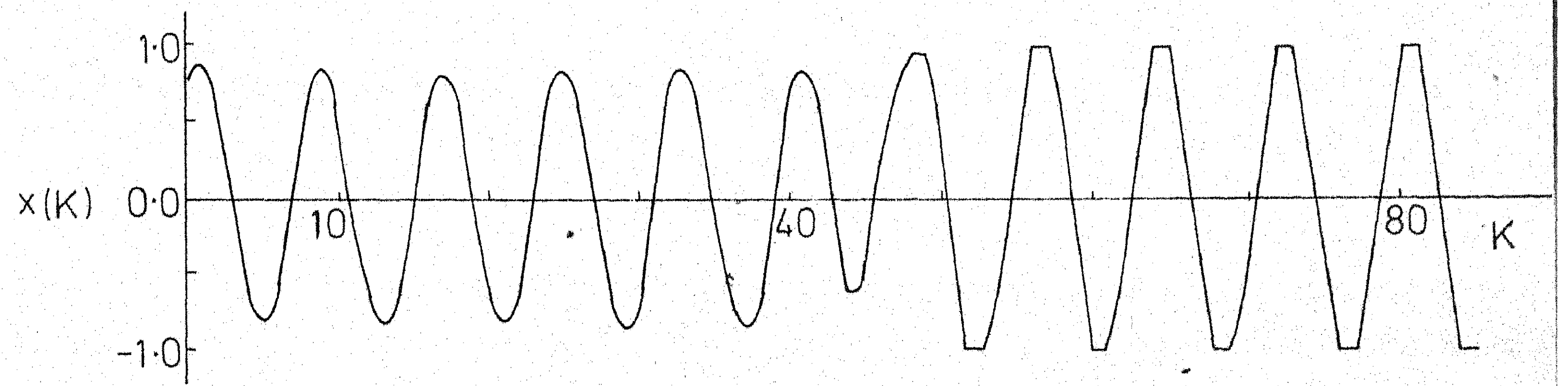


(d) Jump between solutions x(43)=-0·0784, x(44)=0·1165

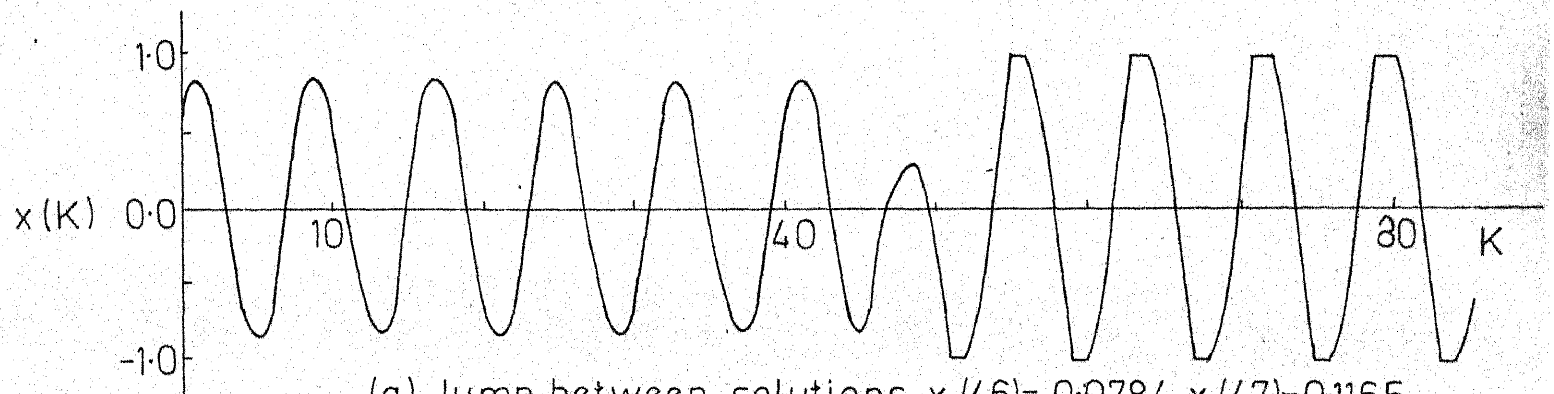
FIG.5·8 SOLUTION TO EQN (5·1) WITH a=1·85, b=-0·95, F=-0·4



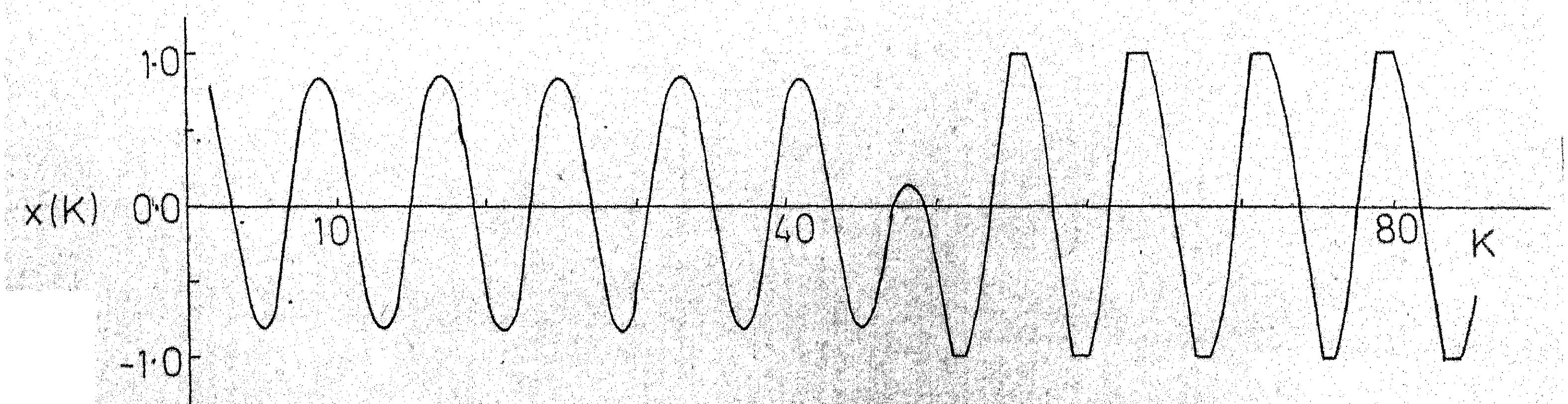
(e) Jump between solutions x(44)=-0·0784, x(45)=0·1165



(f) Jump between solutions x(45)=-0·0784, x(46)=0·1165



(g) Jump between solutions x(46)=-0·0784, x(47)=0·1165



(h) Jump between solutions x(47)=-0·0784, x(48)=0·1165

FIG. 5·8 SOLUTION TO EQN.(5·1) WITH a=1·85, b=-0·95, F=-0·4

The region ABCBA in Fig. 5.3 on the otherhand satisfying C-2 and C-3 alone, sustains nonlinear oscillations. This can be verified by picking coefficient values [a,b] inside this region and simulating the system given in (5.1) for initial values given in (5.6) and (5.7). This response is shown in Fig. 5.9. The nonlinear sustained oscillation is observed without any jump for all values of k with small initial transient. Fig. 5.10 provides the time response of nonlinear sustained oscillation for parameter values selected from the region where such oscillations are possible for N=4.

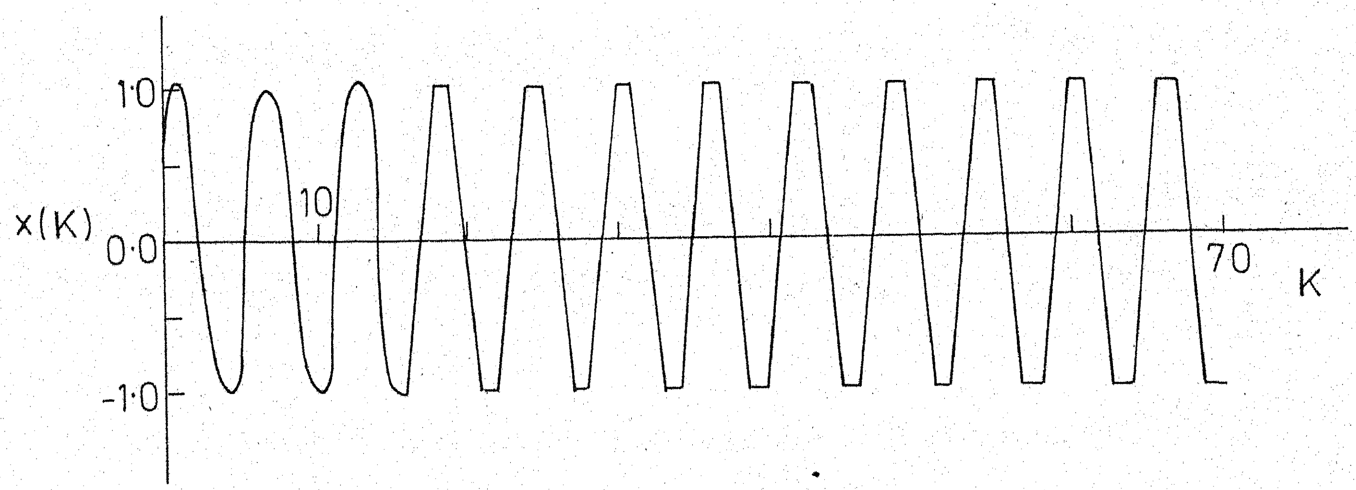
As mentioned earlier the range of input amplitude F for which there exists a jump can be obtained as follows. For specified filter coefficients [a,b] and for input frequency $\alpha = \pi/N$, the value of F satisfying all the three inequalities C-1, C-2 and C-3 gives the required range. For example for $N = 3$, $a = 1.95$, $b = -0.995$, the F satisfying the C-1, C-2 and C-3 conditions in eqn. (5.18) are

$$F > -0.95251$$
$$F > -1.27105$$
$$F < -0.76214$$

respectively.

Then the range of F satisfying all the above three inequalities is given by

$$-0.76214 < F < -0.95251. \tag{5.42}$$



FIG·5·9 a=1·45, b=-0·95, F=-0·5, N=3
(Equation 5·1)

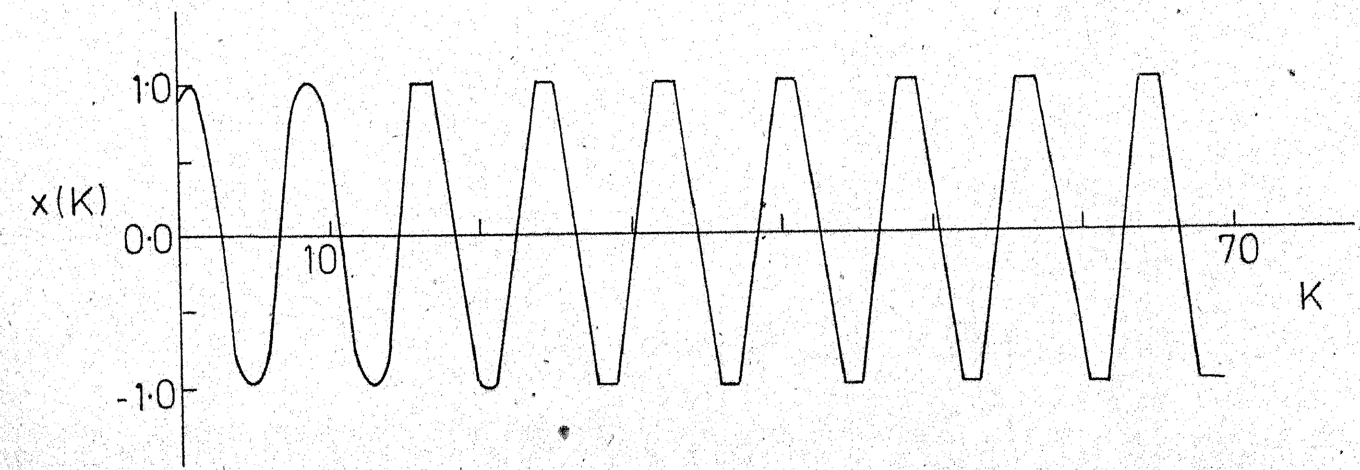


FIG.5·10 a=1·7, b=-0·875, F=-0·4, N=4
(Equation 5·1)

Selecting F = -0.8 (which is within the range given in eqn. (5.42)) the time response is given in Fig. 5.11. Fig. 5.11a shows the linear response with $x(o) = 0.426$ and $x(1) = 0.840$, Fig. 5.11b gives the nonlinear oscillation for $x(o) = x(1) = 1.0$ and Fig. 5.11c shows the jump between the solutions.

Then for N = 4, a = 1.95, b = -0.99, F satisfying the conditions given in eqn. (5.22) are

$$F > -0.5429$$
$$F > -0.5462$$
$$F < -0.4222$$

from which the range of F satisfying all the above inequalities is given by

$$-0.5429 < F < -0.4222$$

The plot shown in Fig. 5.12 gives the time response of the system for the coefficient values given above and with F = -0.446. Here again the jump up behaviour between two steady state solutions is evident.

Similarly for N = 5, a = 1.9,b = -0.95, the range of F is given by

$$-0.3116 < F < -0.2746$$

and for N = 6, a = 1.9, b = -0.95, the range of F is given by

$$-0.2032 < F < -0.1889$$

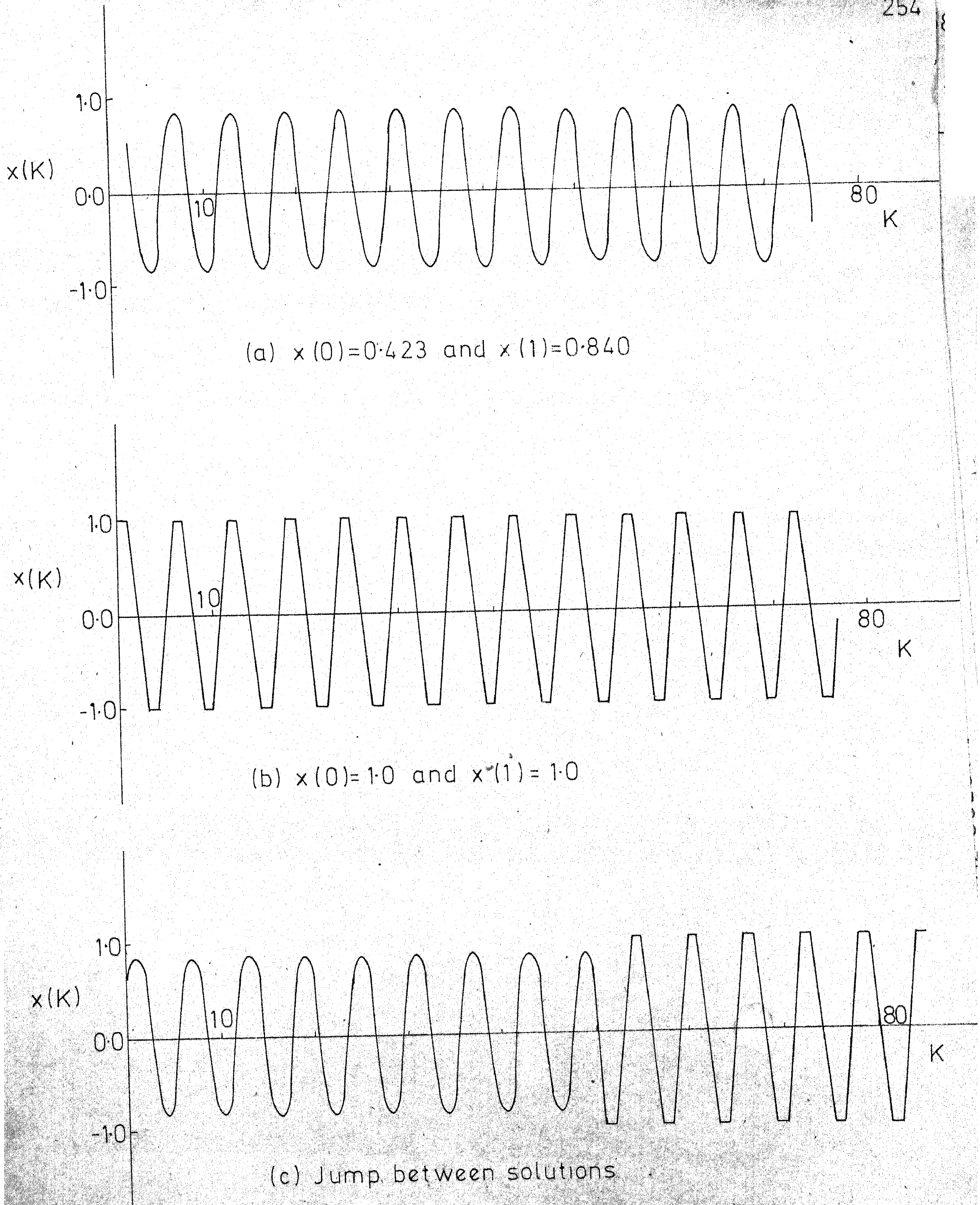
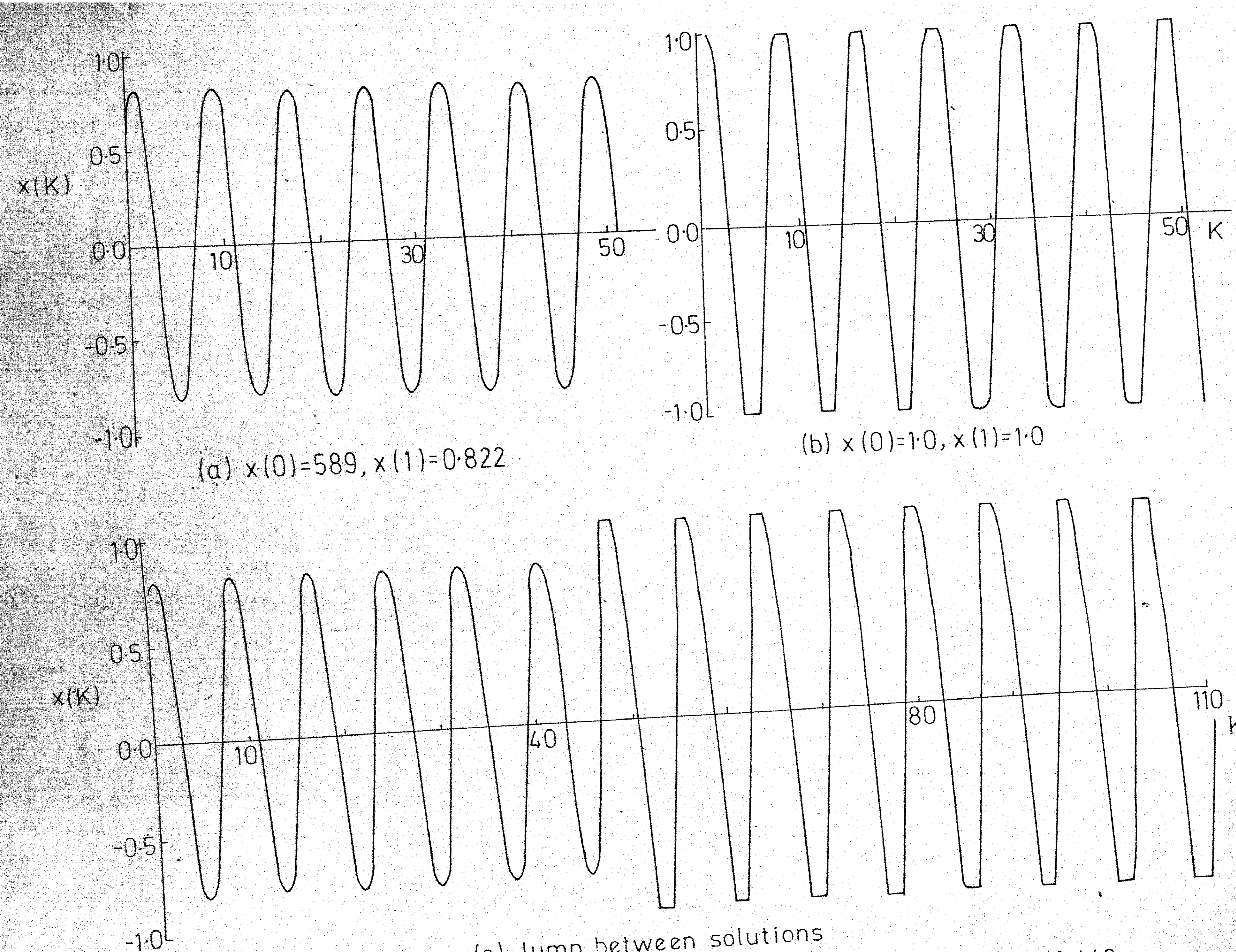


FIG.5.11 SOLUTION TO EQN. (5.1) WITH $a=1{\cdot}95$, $b=-0{\cdot}995$, $F=-0{\cdot}8$



(a) x(0)=589, x(1)=0·822

(b) x(0)=1·0, x(1)=1·0

(c) Jump between solutions

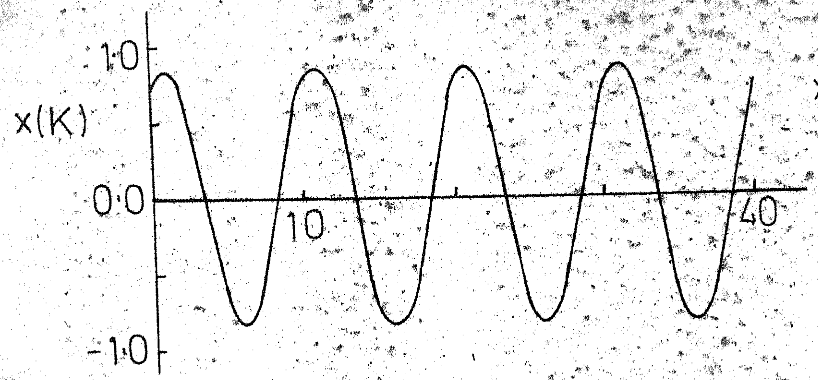
FIG.5.12 SOLUTION TO EQUATION (5.1) WITH a=1·95, b=-0·99, F=-0·446

Figs. 5.13 and 5.14 provide the time plot indicating the jump for $F = -0.274$ and $F = -0.188$ for $N = 5$ and $N = 6$ respectively.

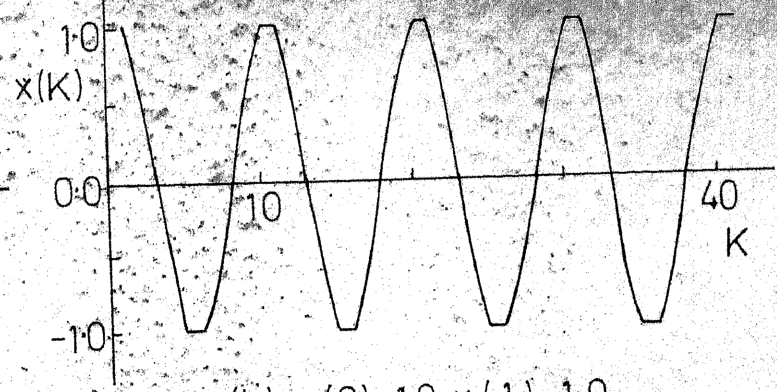
5.5.2 Jump down phenomena:

In the previous section a jump up phenomenon between two steady state solutions is observed in digital filters under forced situation. Jump down phenomenon is also possible under the conditions discussed below.

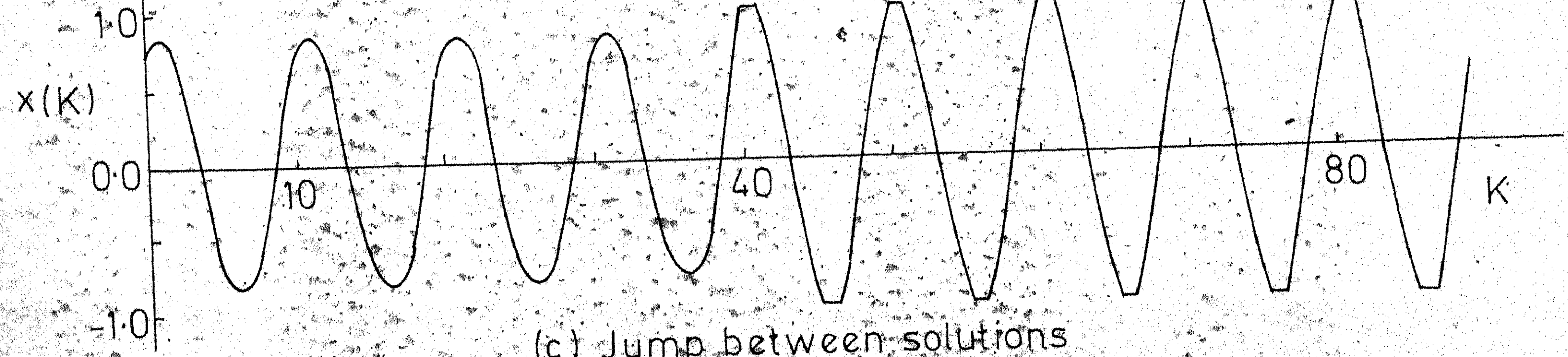
In this case the solution is initially obtained for $x(o) = x_0$ and $x(1) = x_1$ which is periodic with unity magnitude (nonlinear oscillatory solution). When the conditions given in eqns. (5.6) and (5.7) are applied such that $x(j) = x(o)$ and $x(j+1) = x(1)$, where j is the discrete time instant given by $j = 2nN$ for $n = 1,2,3, \ldots.$, a jump down phenomena is observed without transient as shown in Fig. 5.15a for $N = 3$. On the otherhand when the conditions $x(j) = x(o)$ and $x(j+1) = x(1)$ are applied at time instants other than the above, the output sequence shoots up to nonlinear mode after a few cycles of transient behaviour. This is indicated in Figs. 5.15b to 5.15e for $N = 3$ with disturbance application at various time instants as indicated therein. Similar behaviour is observed for other frequences namely $N = 4,5$ and 6.



(a) x(0)=0·727, x(1)=0·845

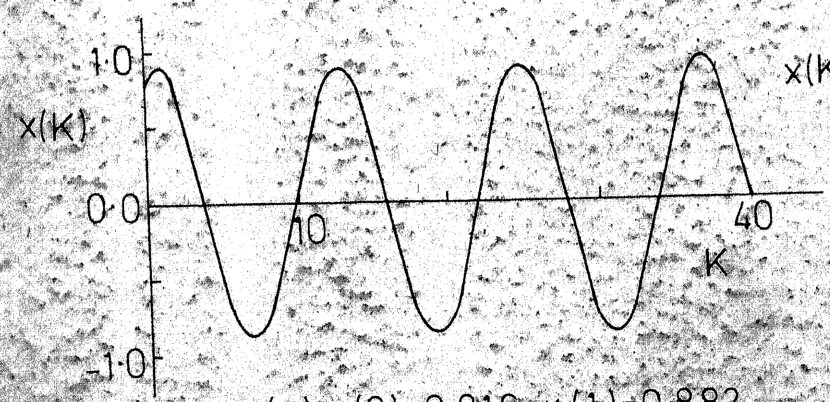


(b) x(0)=1·0, x(1)=1·0

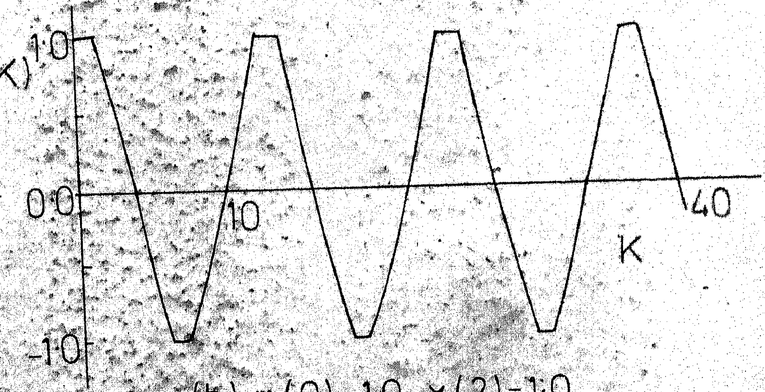


(c) Jump between solutions

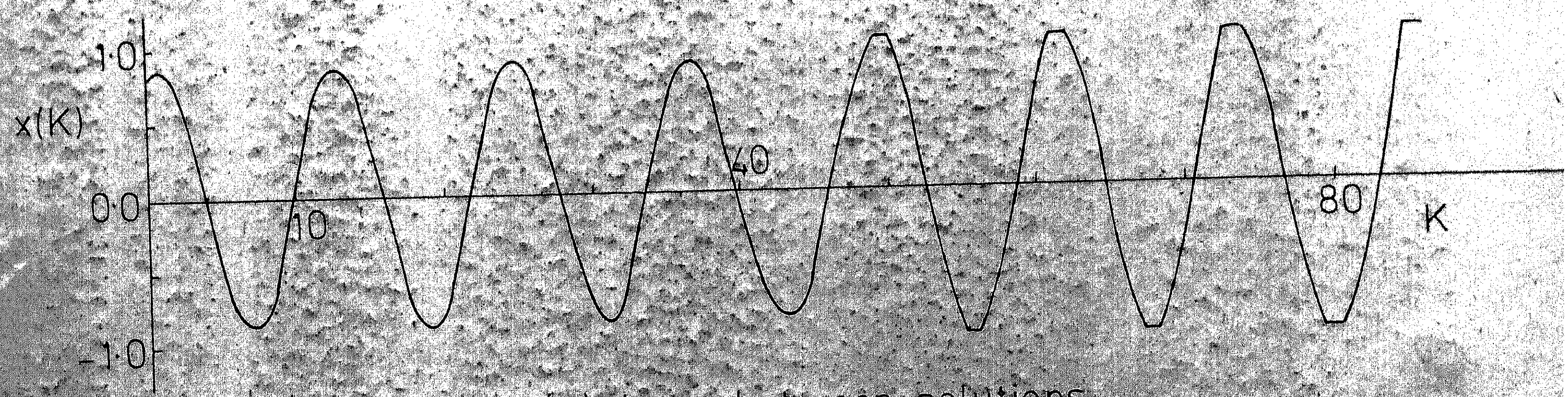
FIG. 5·13 EQN. (5·1) WITH a=1·9, b=0·95, F=-0·274, N=5



(a) x(0)=0·816, x(1)=0·882

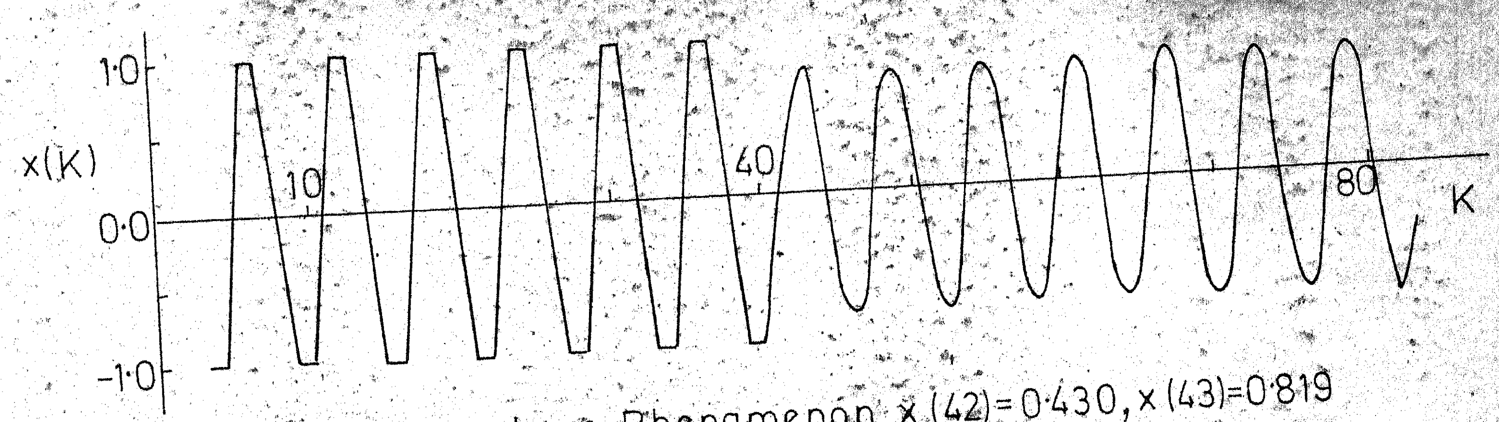


(b) x(0)=1·0, x(2)=1·0

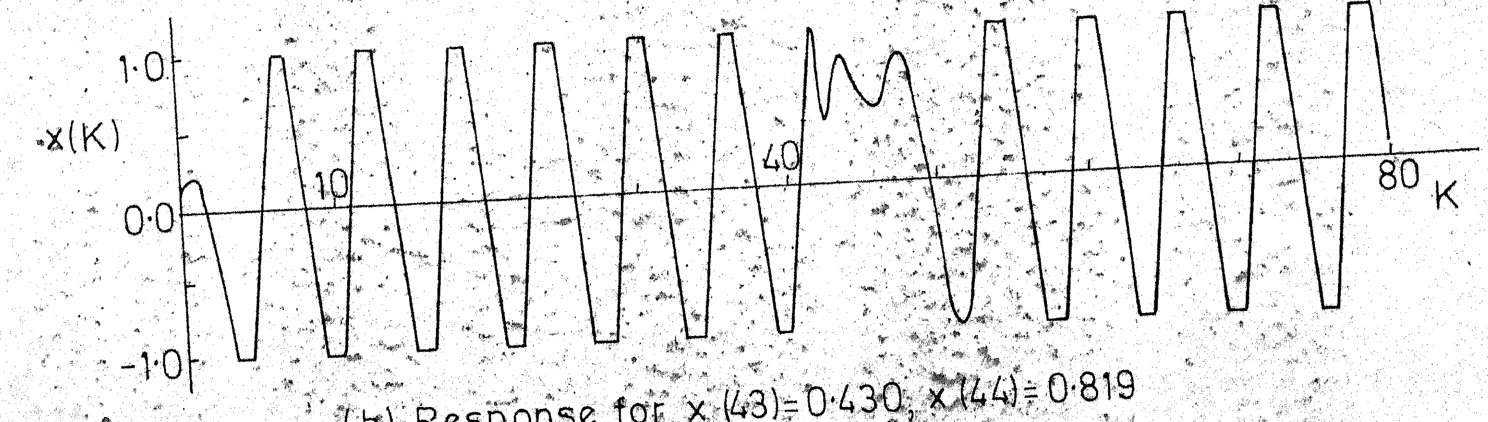


(c) Jump between solutions

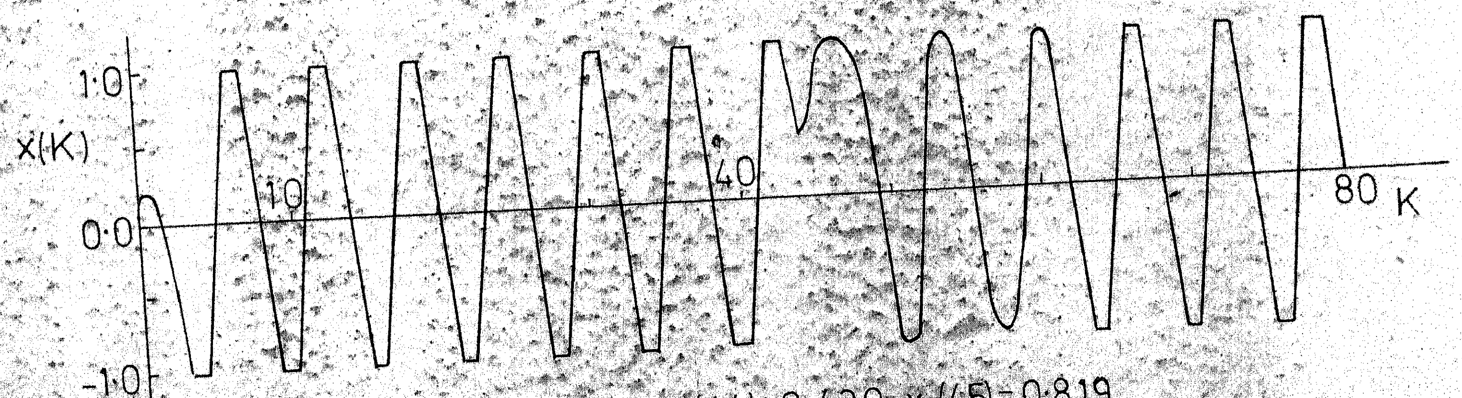
FIG. 5·14 EQN. (5·1) WITH a=1·9, b=0·95, F=-0·188, N=6



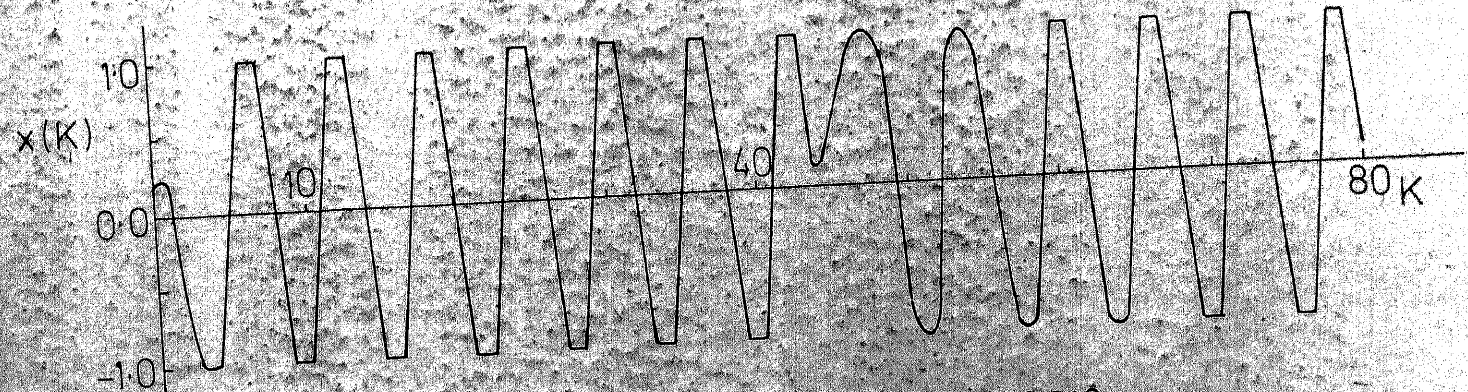
(a) Jump down Phenamenon $x(42)=0{\cdot}430$, $x(43)=0{\cdot}819$



(b) Response for $x(43)=0{\cdot}430$, $x(44)=0{\cdot}819$



(c) Response for $x(44)=0{\cdot}430$, $x(45)=0{\cdot}819$



(d) Response for $x(45)=0{\cdot}430$ $x(46)=0{\cdot}819$

FIG. 5.15 EQN. (5.1) $a=1{\cdot}6$, $b=-0{\cdot}98$, $F=-0{\cdot}5$, $x(0)=0{\cdot}152$, $x(1)=0{\cdot}182$

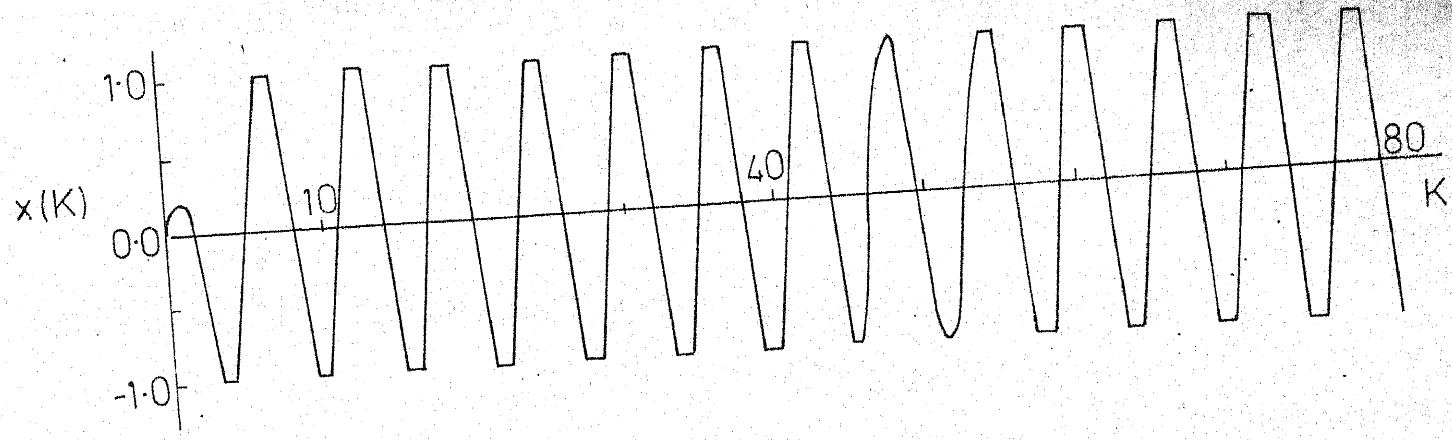


FIG.5·15e RESPONSE FOR x (46)=0·430, x (47)=0·819

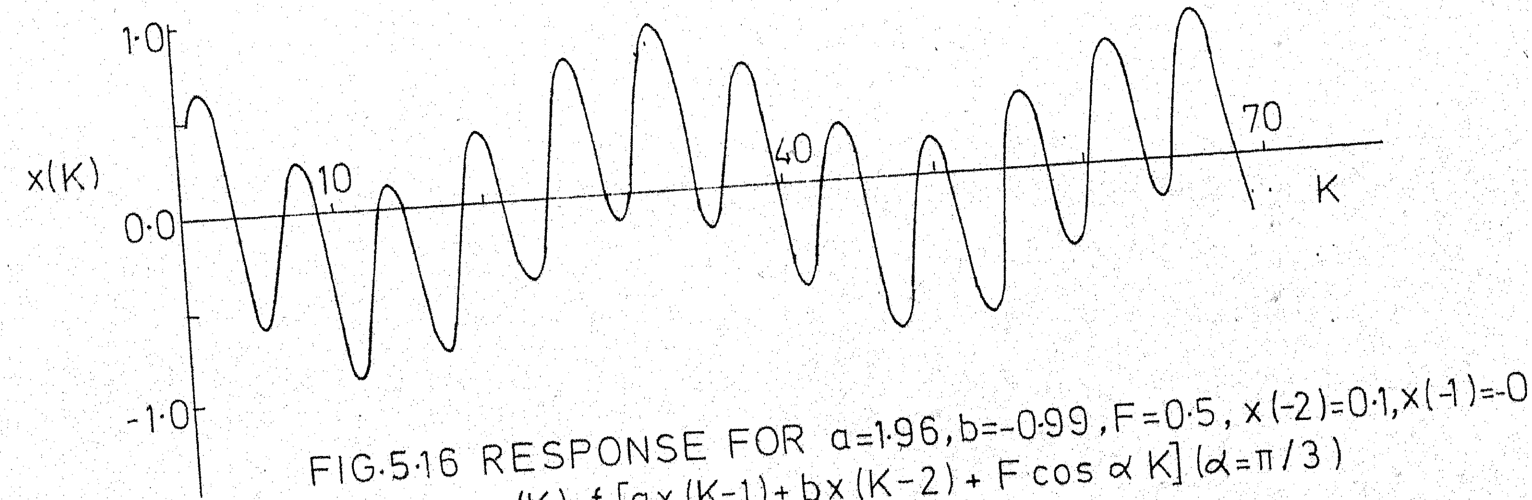


FIG.5·16 RESPONSE FOR $a=1{\cdot}96, b=-0{\cdot}99, F=0{\cdot}5, x(-2)=0{\cdot}1, x(-1)=-0$
$x(K)=f\,[a\,x(K-1)+b\,x(K-2)+F\cos\alpha K]\ (\alpha=\pi/3)$

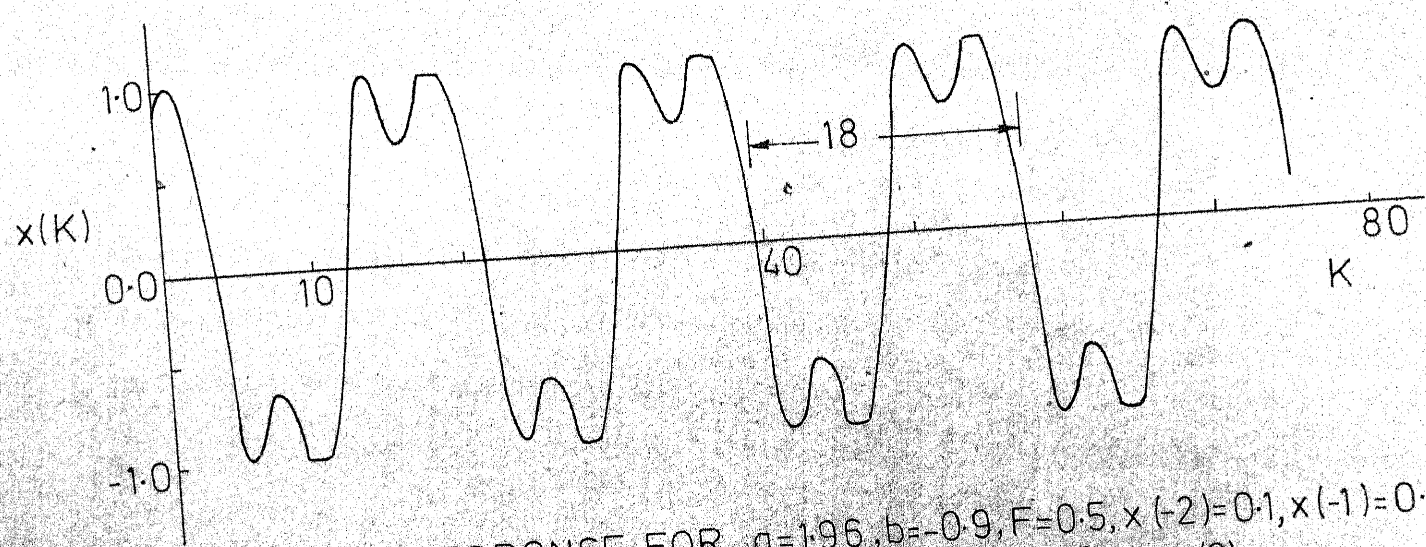


FIG.5·17 RESPONSE FOR $a=1{\cdot}96, b=-0{\cdot}9, F=0{\cdot}5, x(-2)=0{\cdot}1, x(-1)=0{\cdot}9$
$x(K)=f\,[a\,x(K-1)+b\,x(K-2)\ \ F\cos\alpha K]\ (\alpha=\pi/3)$

It is of interest to refer to a similar although less complete investigation due to Kristiansson [95] concerning the nonlinear oscillations in digital filters. As one readily sees, the proposed technique provides information regarding the nonlinear oscillations only for positive values of the parameter 'a'. For negative values of 'a' no jump phenomenon of the kind described is observed. This is, because there exists no F to satisfy the general necessary conditions C-1 to C-3 for specified N. In [95] it has been demonstrated the existence of jumps for small positive values of F (for N = 3), where as in the proposed analysis the admissible value of F turns out to be negative for N = 3 as seen in Fig. 5.3. This is because of a difference in the system model in [95] introducing an additional phase shift in input forcing function.

In the next section the occurrence of super/sub harmonic oscillations in a forced second order digital filter is discussed.

5.6 Super/sub harmonic oscillations :

In nonlinear systems, the steady state response to a harmonic signal may have a frequency which is multiple or submultiple of input frequency and they are in general known as super and sub harmonic components. This kind of phenomenon in a nonlinear discrete time systems has been studied in

detail in chapter three for polynomial type of nonlinearity. In continuous time systems the super/sub harmonic oscillations have been discussed extensively in many text books and research papers. For the present study the available methods can not be extended for studying the super/sub harmonic oscillations in nonlinear digital filters due to discontinuous nature of the nonlinearity. Attempts, however unsuccessful, were made to investigate the super/sub harmonic oscillations in non-linear digital filter with different known techniques, namely the harmonic balance method and two variable expansion procedure. Then the possibility of occurrence of super/sub harmonic oscillations are deducted based on exhaustive simulation of given system. Two kinds of sub harmonic oscillations are observed and the first one is the type, which has been reported by Claasen and Kristiansson [96], the time response of such oscillations shows that the sub harmonic components die out for large time. whereas the second type of sub harmonic oscillation is a steady state oscillation. Figs. 5.16 and 5.17 give these oscillations. For these investigations the system description is same as given in [96].

The existence of nonlinear behaviour such as limit cycles, jump phenomenon and super/sub harmonic oscillations are generally not desirable in designing digital filters [113]. It is important therefore to find methods that avoid

as much as possible, the occurrence of nonlinear phenomena. Claasen et al [96] and Willson [97] have proposed error feedback circuits to improve the behaviour of digital filters minimising the nonlinear effects. It has been shown by them that the proposed feedback circuit do not completely eliminate the occurrence of overflow phenomena, but with optimal choice of feedback coefficients the nonlinear effects are reduced to a minimum.

5.7 Conclusion :

A second order digital filter with overflow saturation nonlinearity is analysed under forced situation. Some conditions on the filter coefficients [a,b] are obtained for given input frequency interms of the input amplitude for which the filter exhibits nonlinear overflow oscillations, such as nonlinear sustained oscillations and jump phenomena. A simple graphical construction of the derived conditions gives the region in the a-b plane (for specified input amplitude) in which either sustained oscillation or jump behaviour is possible. The range for the input amplitude is obtained for specified input frequency and parameter [a,b] values for which the nonlinear behaviour is observed. The occurrence of subharmonic oscillations are studied through exhaustive search by simulating the system equation. The theoretical results obtained are verified by computer simulation.

CHAPTER 6

CONCLUSION

In this thesis a detailed analysis of a class of nonlinear difference equations arising generally in discrete time systems has been carried out. The nonlinear difference equations resulting in a discrete time system are generally of two kinds,

i) Those described by polynomial type of nonlinearity and

ii) Those described by discontinuous type of nonlinearity.

The second type of equations generally exist in the mathematical description of digital filters under nonlinear operation due to wordlength reduction that is necessary on account of finite wordlength registers. In this thesis these two kinds of nonlinear equations have been analysed without and with external excitation. Nonlinear phenomena like limit cycle oscillations under force free condition, jump phenomena and response characteristics with weak forcing and sub/superharmonic solutions under strong input situation are the main investigations.

The difference equations described by polynomial nonlinearities have been analysed adopting Discrete Multiple Scale Perturbational Technique. This technique has already been discussed in depth by Cole [16], Nayfeh [18] and others

for continuous time systems. This technique has been so far applied to study a class of ordinary and partial differential equations as well as a class of third order differential equations [21]. Yen and Kronauer [20] have applied the two time variable expansion procedure to analyse three oscillators coupted by a weak nonlinearity. Kronauer and Musa [23] have studied the possible sub/superharmonic oscillations in a class of second order differential equations using this technique under strong single input situation. Whereas Tiwari and Subramanian [22] have considered the two time scaling technique to investigate super/sub/ultrasubharmonic solutions in a class of nonlinear second order differential equations with strong multiple inputs. The method adapted in this thesis, namely the Discrete Multiple Time Perturbational Technique has been derived based on the properties of finite difference operators.

The proposed method has been tested initially for linear difference equations with under damped, critically damped and unbounded situations and the analytical results are compared with simulation results. For free and weakly forced nonlinear systems, a discrete version of a Duffing oscillator has been considered and the well known response characteristics, loci of vertical and horizontal tangencies have been studied both with and without damping. The existence of limit cycle oscillation has been shown by considering a Van der Pol type

of second order difference equation and the stability analysis of such limit cycles has been carried out through a variational technique. In order to reduce the mathematical calculations involved in the proposed method, a modified definition for the two time discrete model has been provided and the results obtained by this modified technique have been compared with the other approaches.

Strongly driven second order (weakly nonlinear) difference equations have also been considered and generation of sub/superharmonics has been investigated. Both the discrete multiple time perturbational technique and the harmonic balance method have been adapted in the above investigation and the two results compared.

Free second order digital filters with saturation overflow nonlinearity have been considered and the existence of limit cycle oscillations as well as a monotonically decaying type of response for parameter values outside the stability triangle in the parameter plane have been demonstrated. Regions in the parameter space where the limit cycles of different periods are possible and locations both in the parameter plane and in the initial condition plane where the output sequence decreases monotonically have been mapped. In the analysis of limit cycle oscillations three methods have been proposed for locating the various pertinent regions in the parameter plane. The first of these is a variation of one

proposed by Ebert et al [76] whereas the other two are based on a polynomial approximation to the saturation nonlinearity. The third method exhibits the use of discrete multiple time perturbational approach and is believed to be quite novel.

Harmonically forced second order digital filters have also been considered and the existence of jump resonance, super/subharmonic solutions have been studied with saturation overflow nonlinearity. Knowing the input frequency and the filter coefficients, conditions have been derived for the nonlinear oscillations to exist. A range of values for the input amplitude has been obtained for the existence of non-linear oscillations, such as jump phenomena and nonlinear sustained oscillations. The minimum perturbation or disturbance (from a linear oscillatory solution) necessary to cause the above said nonlinear oscillations have also been obtained. The results obtained for jump up and jump down conditions have been verified by computer simulation.

Scope for Further Work :

The problems listed below need further investigations in the study of free and forced oscillations in a class of second order nonlinear difference equations with polynomial as well as discontinuous type of nonlinearities.

1. The Discrete Multiple Time Perturbational Technique proposed in this thesis appears to be nonunique in the sense that the derivation is based on the properties of finite difference operators. It would be worthwhile investigating whether other techniques based on theory of finite difference operators are available for analysis of discrete time systems. An appropriate comparison can be made.

2. It would be interesting to work out some general existence conditions for presence of nonlinear phenomena like limit cycle oscillations, jump phenomena etc.

3. The proposed multiple scale perturbational technique can be extended to study certain higher order as well as coupled nonlinear discrete time systems.

4. The scheme 2, (introduced in section 2.8 of this thesis) namely the modification over the proposed method has been used to obtain more accurate solution for equations whose response is known to be bounded. It would be interesting to investigate in detail as to the reasons for which the modified definition in eqn. (2.90) gives more accurate qualitative results.

5. For force free second order digital filter a region in which a monotonic type of decaying response possible has been investigated. It would be interesting to investigate existence of other kinds of decaying responses for instance, oscillatory decay.

6. In forced second order digital filter with saturation nonlinearity the existence of subharmonic oscillations have been investigated by exhaustive simulation of the system equation. It would be useful to propose an analytical technique for investigating the above phenomena in a discrete system described by discontinuous type of nonlinearities.

REFERENCES

[1] VON KARMAN, TH, The engineering grapples with Nonlinear Problems', Bulletin American Mathematical Society, Vol. 46, pp. 615-683, 1940.

[2] APPLETON, E.V. and VANDER POL, B, 'On the form of free triode vibrations', Philosophical Magazine, Ser.6, Vol. 42, pp. 201-220, 1921.

[3] APPLETON, E.V. and VANDER POL, B, 'On a type of oscillation - Hystensis in a simple generator', Philosophical Magazine, Ser. 6, Vol. 43, pp. 117-193, 1922.

[4] HAYASHI, C, 'Nonlinear oscillations in Physical Systems', McGraw-Hill, New York, 1964.

[5] STOKER, J.J, 'Nonlinear Vibrations', Interscience Publishers', New York, 1950.

[6] CUNNINGHAM, W.J, 'Introduction to Nonlinear Analysis', McGraw-Hill, New York, 1958.

[7] KU, Y.H., 'Analysis and control of nonlinear systems', The Ronald Press Company, New York, 1958.

[8] MCLACHLAN, N.W, 'Ordinary Nonlinear Differential Equations in Engineering and Physical Sciences', Oxford University Press, New York, 1950.

[9] ANDRONOW, A.A. and CHAIKIN, C.E, 'Theory of Oscillations', Princeton University Press, Princeton, NJ, 1949.

[10] GIBSON, J.E, 'Nonlinear Automatic Control', McGraw Hill Book Company, New York, 1963.

[11] VANDER POL, B, 'The Nonlinear Theory of Electric Oscillations', Proc. IRE, Vol. 22, pp. 1051-1086, 1934.

[12] KRYLOV, N.M. and BOGOLIUBOV, N.N, 'Introduction to Nonlinear Mechanics', Mascow, 1937.

[13] BOGOLIUBOV, N.N. and MITROPOLSKI, Y.A, 'Asymptotic methods in the theory of nonlinear oscillations', Gordon and Breach, New York, 1961.

[14] COLE, J.D. and KEVORKIAN, J, 'Uniformly valid asymptotic approximations for certain nonlinear differential equations and nonlinear mechanics', Colorado Springs, Academic Press, New York, 1963.

[15] KEVORKIAN, J, 'The two variable expansion procedure for the approximate solution of certain nonlinear differential equations', Scattle, University of Washington, 1965.

[16] COLE, J.D, 'Perturbation methods in applied mathematics', Waltham, Massachusetts, Blaisdell Publishing Company', 1968.

[17] CARRIER, G.F. and PEARSON, E.E, 'Ordinary Differential Equations', Blaisdell Publishing Company, Waltham, Mass., 1968.

[18] NAYFEH, A.H, 'Perturbation Methods', John Wiley, 1973.

[19] RASMUSSEN, M.L, 'Uniformly valid approximations for non-linear oscillations with small damping', Int. J. Non-linear Mechanics, Vol. 5, pp. 687-696, 1970.

[20] YEN, N. and KRONAUER, R.E, 'Response of three oscillators coupled by a weak nonlinearity', J. of applied Mechanics, Vol. 44, Trans. ASME, Vol. 99, Ser. E, pp. 141-146, 1977.

[21] TIWARI, R.N. and SUBRAMANIAN, R, 'Multiple time scaling for analysis of third order nonlinear differential equations', J. of Sound and Vibration, 52(2), pp. 165-169, 1977.

[22] TIWARI, R.N. and SUBRAMANIAN, R, 'Subharmonic and super-harmonic synchronisation in weakly nonlinear systems', J. of Sound and Vibration, 47(4), pp. 501-508, 1976.

[23] KRONAUER, R.E. and MUSA, S.A, 'Necessary conditions for subharmonic and superharmonic synchronization in weakly nonlinear systems', Quarterly of Appl. Math., XXIV, pp. 153-1960, 1966.

[24] LICK, W, 'Two variable expansions and singular perturbation problems', SIAM. J. APP. Math., Vol. 17, pp. 815-825, 1969.

[25] JORDAN, V.S, 'Two Timing Solution of Mathieu Equation of Second Order', Quar. App. Math., P. 147, 1973.

[26] DASARATHY, B.V. and SRINIVASAN, P, 'Study of class of time dependent systems', Int. J. Control, Vol. 6, No.4, pp. 337-342, 1967.

[27] JONES, S.E. and AMES, W.F, 'Similarity variables and first integrals of Ordinary Differential Equations', Int. J. Nonlinear Mechanics, Vol. 2, pp. 257-260, 1967.

[28] MUSA, S.A. and SRIVASTAVA, S, 'Infinitesimal transformations for nonlinear time varying systems', Int. J. Control, Vol. 10, No.5, pp. 583-592, 1969.

[29] PORTER, W.A, 'Modern Foundations on System Engineering', The Macmillan Company, New York, 1966.

[30] HOLTZMAN, J.M, 'Nonlinear system theory, A functional analysis approach', Prentice Hall, Inc., New York, 1970.

[31] MILNE and THOMSON, 'The calculus of finite differences', The Macmillan, London, 1933.

[32] LEVY, H. and LESSMAN, F, 'Finite Difference Equations', Macmillian, New York, 1961.

[33] JORDAN, C. and CARVER, H.C, 'Calculus of finite differences' Chelsea, New York, 1950.

[34] JURY, E.I. and TSYPKIN, Y.Z, 'On the theory of discrete systems', Automatica, Vol. 7, pp. 89-107, 1971.

[35] HSU, C.S. and YEE, H.C, 'Behaviour of dynamical systems governed by a simple nonlinear difference equations', J. App. Mechanics, Vol. 42, Trans. ASME, Vol. 97, Ser. E, pp. 870-876, 1975.

[36] LORENZ, E.V, 'The problem of deducing the climate from the governing equations', Tellus, Vol. 16, pp. 1-11, 1964.

[37] MAY, R.M, 'Biological populations with nonoverlapping generations, stable points, stable cycles and chaos', Science, Vol. 186, pp. 645-647, 1974.

[38] HSU, C.S. and CHENG, W.H, 'Application of the theory of impulsive parametric excitation and new treatments of general parametric excitation problems', J. app. Mechanics, Vol. 40, Trans. ASME, Vol. 95, Ser. E, pp. 78-86, 1973.

[39] STANLEY, W.D, 'Digital signal processing', Reston Publishing Company, WC, Reston, Virginia, 1975.

[40] LIU, B, 'Effects of finite wordlength on the accuracy of digital filters - A review', IEEE Trans. Circuit Theory, Vol. CT-18, No.6, pp. 670-677, 1971.

[41] OPPENHEIM, A.V. and WEINSTEIN, C.J, 'Effects of finite register length in Digital Filtering and Fast Fourier Transform', Proc. IEEE, pp. 957-976, 1972.

[42] PARKER, S.R, 'The phenomena of quantization error and limit cycles in fixed point digital filters', Proc. National Electronic Conference, Chicago, Vol. 27, pp. 38-42, 1972.

[43] SANDBERG, I.W. and KAISER, J.F, 'A bound on limit cycles in fixed point implementation of digital filters', IEEE Trans. Audio Electroacoust. Vol. AU-20, pp. 110-112, 1972.

[44] MACCOLL, C.A, 'Fundamental theory of servomechanisms', D. Van Nostrand, Princeton, 1945.

[45] JAMES, H, NICHOLS, N.B. and PHILIPS, R.S, 'Theory of Servomechanisms', McGraw-Hill, New York, 1947.

[46] OLDENBERG, R.C. and SARTORIUS, H, 'The dynamics of Automatic Control', ASME, New York, 1948.

[47] KALMAN, R.E. and BERTRAM, J.E, 'Control system analysis and design via second method of Liapunov, Part II, Discrete Time systems', J. of Basic Engineering, Vol. 82, pp. 394-400, 1960.

[48] HURT, T, 'Some stability theorems for ordinary difference equations', J. of Num. Anal., Vol. 4, No.4, pp. 582-596, 1967.

[49] SINHA, A.S.C, 'On stability of class of discrete systems', J. Franklin Institute (USA), Vol. 297, No.4, pp. 287-292, 1974.

[50] WEAVER and LEAKE, 'Stability and boundedness domain of autonomous discrete time systems', Presented at the ASME Winter Annual 70WA/Aug. 12, 1970.

[51] DUTCHAK, B.I. and SINSIAILY, L.A, 'On Liapunov's stability theorems for difference equations', Soviet Auto. Control (USA), Vol. 6, No.6, pp. 12-15, 1973.

[52] GORDON, S.P, 'A stability theory of perturbed difference equations', SIAM, J. Control, Vol. 10, No.4, pp. 671-678, 1972.

[53] 'Stability of discrete systems with periodic coefficients' 11th Joint automatic control conference of American Automatic control council, ASME, pp. 225-228, 1970.

[54] FREEMAN, H, 'Discrete time analysis' An introduction to the theory, John Wiley, New York, 1965.

[55] CADZOW, J.A, 'Discrete time systems', An introduction with interdisciplinary applications, NJ, Printice Hall, 1973.

[56] RAGAZZINI, J.R. and ZADEH, L.A, 'The analysis of sampled data systems', Trans. AIEE, Vol. 71, Part II, pp. 225-234, 1952.

[57] BARKER, R.H, 'The pulse transfer function and its application to sampling servo systems in frequency response', Macmillan Company, New York, 1956.

[58] KUO, B.C, 'Analysis and synthesis of sampled data systems', Printice Hall, Englewood Cliffs, NJ, 1963.

[59] JURY, E.I, 'Theory and application of Z-transform method', John Wiley and Sons, New York, 1964.

[60] LINDORFF, D.P, 'Theory of sampled data control systems', John Wiley, New York, 1965.

[61] SEDLAR, M. and BAKEY, G.A, 'Signal flow graphs of sampled data systems - a new formulation', IEEE Trans. Aut. Control, Vol. AC-12, pp. 154-161, 1967.

[62] TRONG, H.C, 'Second order nonlinear difference equation containing small parameter', J. of Franklin Institute, Vol. 269, pp. 97-101, 1960.

[63] HUSTON, R.L, 'Krylov-Bogoliubov's method for difference equations', SIAM. Appl. Maths., Vol. 19, No.2, pp 334-339, 1970.

[64] PAI, M.A. and JURY, E.I, 'Convolution Z-transform applied to certain nonlinear discrete systems', IRE. Trans. PGAC, Vol. AC-7, No.1, pp. 57-64, 1962.

[65] REDDY, B.E. and JAGAN, N.C, 'Evaluation of response of nonlinear discrete systems using multidimensional Z-transforms', Proc. IEE, Vol. 119, No. 10, pp. 1521-1525, 1972.

[66] LAVI, A. and NARAYAN, S, 'Analysis of a class of nonlinear discrete systems using multidimensional modified Z-transform, 'IEEE Trans. Aut. Control, Vol. AC-13, pp. 90-92, 1968.

[67] JAGAN, N.C. and DESAI, R.C, 'Analysis of nonlinear time varying discrete systems by the use of multidimension Z-transforms', Proc. IEE (G.B.), Vol. 121, No.4, P307, 1974.

[68] ALPER, P, 'A consideration of the discrete Volterra series', IEEE Trans. Aut. Control, Vol. AC-10, pp. 322-327, 1965.

[69] FU, R.C, 'Analysis of a class of nonlinear discrete time systems by the Volterra series', Int. J. Control, Vol. 18, No.8, pp. 545-551, 1973.

[70] ASELTINE, J.A. and NESBIT, R.A, 'The incremental phase plane for nonlinear sampled data systems', IRE Trans., Aut. Control, Vol. AC-5, pp. 159-165, 1960.

[71] LINDROFF, D.F, 'Use of a coordinate conversion in the incremental phase plane', IEEE Trans., Aut. Control, Vol. AC-9, No.1, pp. 110-113, 1964.

[72] PAI, M.A, 'Oscillations in nonlinear sampled data systems', AIEE Trans., Vol. 81, Part II, pp. 350-355, 1962.

[73] GOLD, B. and RADER, C.M, 'Effects of quantization noise in digital filters', in Proc., 1966 Spring joint Comput. Conf. Proc., Montvale, NJ, AFIPS Press, Vol. 28, pp. 213-219, 1966.

[74] KAISER, J.F, 'The digital filter and speech communication', IEEE Trans. Audio Electroacoust. (Special Issue on speech communication and processing - Part II) Vol. AU-16, pp. 180-183, 1968.

[75] JACKSON, L.B, 'An analysis of limit cycles due to multiplication rounding in recursive digital filters', in Proc. 7th Annual Allerton Conf. Circuit and System Theory, Monticello, IL, pp. 69-78, 1969.

[76] EBERT, P.M, MAZO, J.F. and TAYLOR, M.G, 'Overflow oscillations in digital filters', Bell Syst. Tech. J, Vol. 48, pp. 2999-3020, 1969.

[77] FETTWEIS, A, 'Pseudopassivity, Sensitivity and stability of wave digital filters', IEE Trans. Circuit Theory (Special issue on nonlinear circuits), Vol. CT-19, pp. 668-673, 1972.

[78] FETTWEIS, A, 'Wave digital filters with reduced number of delays', Int. J. of Circuit Theory and applications, Vol. 2, pp. 319-330, 1973.

[79] CLAASEN, T.A.C.M, MECHLENBRAKER, W.F.G. and PEEK, J.B.H, 'Effects of quantization and overflow in recursive digital filters', IEEE Trans. Acoust. speech, Signal Processing, Vol. ASSP-24, No.6, pp. 517-529, 1976.

[80] SANDBERG, 'A theorem concerning limit cycles in digital filters', in Proc. 7th Annu. Allerton Conf. on circuits and system theory, pp. 63-68, 1969.

[81] MONTGOMERY, H.D, 'A nonlinear digital oscillator', Proc. of the IEEE 1972 Int. Conf. on Communications, Philadelphia, Pennsylvania, pp. 33.3-33.7, 1972.

[82] MITRA, D. and MONTGOMERY, H.D, 'On digital oscillator', Proc. tenth Allerton Conf. on Circuits and System Theory, Monticello, Illinois, pp. 709-723, 1972.

[83] WILLSON, A.N, 'A stability criterion for nonautonomous difference equations with application of the design of a digital FSK oscillator', IEEE Trans. circuits and systems, Vol. CAS-21, No.1, pp. 124-130, 1974.

[84] WILLSON, A.N, 'Limit cycles due to adder overflow in digital filters', IEEE Trans. Circuit Theory, Vol. CT-19, pp. 342-346, 1972.

[85] CLAASEN, T.A.C.M, MECHLEMBRAUKER, W.F.G. and PEEK, J.B.H, 'On the stability of the forced response of digital filters with overflow nonlinearities', IEEE Trans. Circuits and Systems, Vol. CAS-22, pp. 692-696, 1975.

[86] CLAASEN, T.A.C.M. MECHLENBRAUKER, W.F.G. and PEEK, J.B.H, 'Some remarks on the classification of limit cycles in digital filters', Philips Res. Rep. Vol. 28, pp. 297-305, 1973.

[87] LONG, J.L. and TRICK, T.N, 'A note on absolute bounds on limit cycles due to round of errors in digital filters', IEEE Trans. Audio Electroacoust.Vol. AU-21, pp. 27-30, 1973.

[88] JATNICKS, G.V. and SHENOI, B.A., 'Zero input limit cycles in coupled digital filters', IEEE Trans. Acoust. speech, Signal Processing, Vol. ASSP-22, pp. 146-149, 1974.

[89] CLAASEN, T.A.C.M, MECHLENBRAUKER, W.F.G. and PEEK, J.B.H, 'Frequency domain criteria for absence of zero input limit cycles in nonlinear discrete time systems with application to digital filters', IEEE Trans. circuits and systems, (Special issue on digital filtering and image processing) Vol. CAS-22, pp. 232-239, 1975.

[90] FETTWEIS, A. and MEERKOTTER, K, 'Suppression of parasitic oscillations in wave digital filters', IEEE Trans. Circuits and systems (Special issue on digital filtering and image processing), Vol. CAS-22, pp. 239-246, 1975.

[91] WILLSON, A.N, 'Computation of periods of forced overflow oscillations in digital filters', IEEE Trans. Acoust, speeck, Singral Processing, Vol. ASSP-24, pp. 89-97, 1976.

[92] MITRA, D, 'Large amplitude, self sustained oscillations in difference equations which describe digital filter sections using saturation arithmetic', IEEE Trans. Acoust.,Speech Singral processing, Vol. ASSP-25, No.2, pp. 134-143, 1977.

[93] PARKER, S.R. and HESS, R, 'Limit cycle oscillations in digital filters', IEEE Trans. Circuit Theory, Vol. CT-18, No.6, pp. 687-697, 1971.

[94] WILLSON, A.N, 'Some effects of quantization and overflow on the forced response of digital filters', Bell Syst. Tech. J. Vol. 51, pp. 863-887, 1972.

[95] KRISTIANSSON, L.O.G, 'Jump phenomena in digital filters', Electron Lett., Vol. 10, pp. 14-15, Jan. 24, 1974.

[96] CLAASEN, T.A.C.M. and KRISTIANSSON, L.O.G, 'Improvement of overflow behaviour of second order digital filter by means of error feedback', Electron Lett., Vol. 10, pp. 240-241, June 13, 1974.

[97] WILLSON, A.N, 'Error feedback circuits for digital filters', Electron Lett., Vol. 12, No. 18, pp. 450-452, Sep. 2, 1976.

[98] KAISER, J.F, 'On the limit cycle problems', in Proc. IEEE Int. Symp. Circuits and Systems, Munich, Germany, pp. 642-645, 1976.

[99] SUBRAMANIAN, R. and KRISHNAN, A, 'Nonlinear discrete time systems analysis using multitime perturbation technique', to be published in J. of Sound and Vibration, April, 1979.

[100] KRISHNAN, A. and SUBRAMANIAN, R, 'Difference equations in nonlinear discrete time systems', In Proc. 4th National system conference (NSC 77), held at Coimbatore, June 1977.

[101] HOPPENSTEADT, F.C. and MIRANKER, W.L, 'Multitime method for system of difference equations', J. Studies in applied Mathematics, Vol. 56, pp. 273-289, 1977.

[102] KRISHNAN, A. and SUBRAMANIAN, R, 'Limit cycles in digital filters', In Proc. 5th National systems conference (NSC78), held at Ludhiana, Sep. 1978.

[103] KRISHNAN, A. and SUBRAMANIAN, R, 'Jump phenomena in digital filters', Communicated to the J. Electronics and Telecommunication Engineers (India).

[104] MORRISON, J.A, 'Comparison of the modified method of averaging and the two variable expansion procedure', SIAM Review, Vol. 8, No.1, pp. 66-85, 1966.

[105] REISS, E.L., 'On multivariable asymptotic expansion', SIAM Review, Vol. 13, pp. 189-196, 1971.

[106] LOVINE and OBI, W.C, 'Multivariable expansion using time scales generated from the differential equation', SIAM J. Appl. Math. Vol. 33, No.4, p.581, 1977.

[107] ROSCOE, D.F, 'New methods for the derivation of stable difference representations for differential equations', J. Inst. Maths. Applies, Vol. 16, pp. 291-301, 1975.

[108] SMITH, G.D, 'Numerical solutions of partial differential equations', Oxford Press, 1965.

[109] HILDEBRAND, F.B, 'Finite difference equations and simulations', Prentice Hall, 1968.

[110] RABINER, L.R. et al, 'Terminology in digital signal processing', IEEE Trans. Audio Electroacoust., Vol. AU-20, pp. 322-337, 1972.

[111] YAKOWIZ, S. and PARKER, S.R, 'Computation of bounds for digital filter quantization errors', IEEE Trans., Circuit Theory, Vol. CT-20, pp. 391-396, 1973.

[112] CLAASEN, T.A.C.M. and KRISTIANSSON, L.O.G, 'Necessary and sufficient conditions for the absence of overflow phenomena in a second order recursive digital filters', IEEE Trans. Acoust,Speech, Signal Processing, Vol. ASSP-23, pp. 509-515, 1975.

[113] KANEKO, T, 'Limit cycle oscillations in floating point Digital filters', IEEE Trans. Audio and Electroacoust, Vol. AU-21, pp. 100-106, 1973.

[114] TOMAS, J, 'Ultra subharmonic resonance in a duffing system', Int. J. of Nonlinear Mechanics, No.6, pp. 625-631, 1971.

[115] STANISIC, M.M. and EULER, J.A, 'General perturbational solution of a harmonically forced nonlinear oscillator equation', Int. J. Nonlinear Mechanics, Vol. 8, p. 523, 1973.

[116] RIGANTI, R, 'Subharmonic solutions of the Duffing equation with large nonlinearily', Int. J. Nonlinear Mechanics, Vol. 13, pp 21-31, 1978.

[117] MUSA, S.A. and KRONAUER, R.E, 'Sub and super harmonic synchronization in weakly nonlinear systems'. Integral constraints and duality', Quarterly of Appl. Math., Vol. 25, pp. 399-414, 1967.

APPENDIX A

PROPERTIES OF THE STABILITY TRIANGLE

It is well known that for a linear second order difference equation,

$$x(k+2) + ax(k+1) + bx(k) = 0 \qquad (A.1)$$

the stability of the response x(k) can be studied knowing the location of the system parameters [a,b] in the a-b parameter plane (shown in Fig. A.1). When [a,b] values are inside the triangle ACG, the system response is asymptotically stable (decaying to zero), when they are on the sides of the triangle ACG, the solution x(k) is generally bounded (periodic) and for the parameter location outside the triangle ACG the response is always unstable (unbounded response). The triangle ACG is known as 'the stability triangle'.

It is to be noted that the nature of the response of system in eqn. (A.1) depends on the location of [a,b] values in the parameter plane. The response x(k) for different location of system parameters are given below.

(1) at A,

$x(k) = A + Bk$, unbounded response without oscillation.

(2) at C,

$x(k) = (A+Bk)\ (-1)^k$, unbounded oscillatory response.

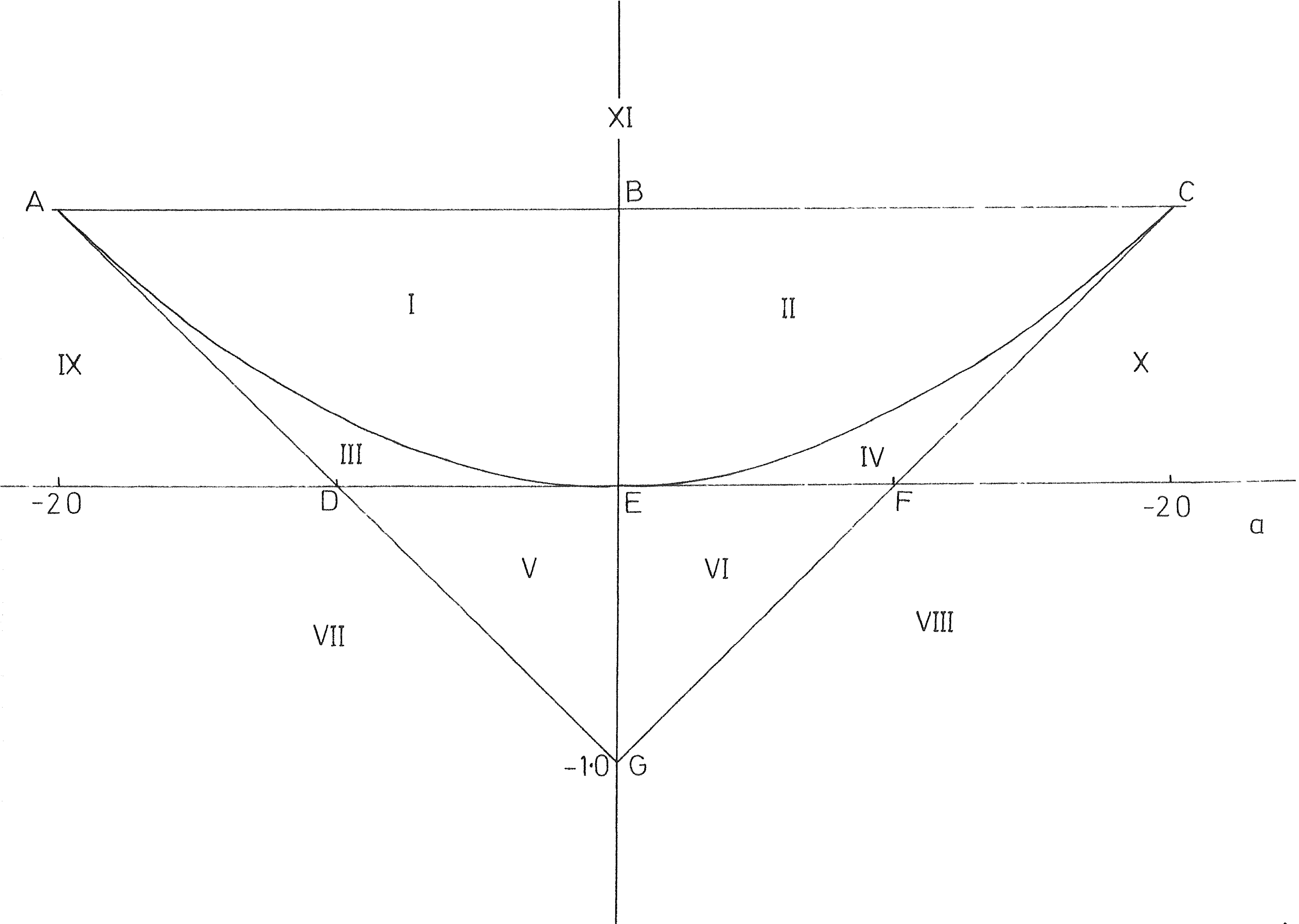


FIG A1 STABILITY TRIANGLE

(13) along the curve CE,

$$x(k) = (A+Bk)\left(-\frac{a}{2}\right)^k,$$ oscillatory decay.

(14) region I,

$$x(k) = (\sqrt{b})^k\,(A\cos\theta_3 k + B\sin\theta_3 k),$$ oscillatory decay.

where $\theta_3 = \cos^{-1}(a/2\sqrt{b})$.

(15) region II,

$$x(k) = (\sqrt{b})^k\,(A\cos\theta_4 k + B\sin\theta_4 k),$$ oscillatory decay.

where $\theta_4 = \pi - \cos^{-1}(a/2\sqrt{b})$.

(16) region III,

$$x(k) = A(\alpha)^k + B(\beta)^k,$$ decays to zero solution

α and β are positive, less than unity.

(17) region IV,

$$x(k) = A(-\alpha)^k + (-\beta)^k,$$ oscillatory decay.

α and β are positive, less than unity.

(18) region V,

$$x(k) = A(\alpha)^k + 3(-\beta)^k,$$ decays to zero solution

α and β are positive, less than unity, $\alpha > \beta$.

(19) region VI,

$$x(k) = A(-\alpha)^k + B(\beta)^k,$$ oscillatory decay.

α and β are positive, less than unity, $\alpha > \beta$.

(20) region VII

$$x(k) = A(\alpha)^k + B(-\beta)^k,$$ unbounded response.

$\alpha > 1.0, \quad 0 > \beta > 1.0.$

(21) region VIII,

$x(k) = A(-\alpha)^k + B(\beta)^k$, unbounded oscillatory.

$\alpha > 1.0, \quad 0 > \beta > 1.0.$

(22) region IX,

$x(k) = A(\alpha)^k + B(\beta)^k$, unbounded response.

$\alpha > 1.0, \quad 0 > \beta > 1.0.$

(23) region X,

$x(k) = A(-\alpha)^k + B(-\beta)^k$, unbounded oscillatory.

$\alpha > 1.0, \quad 0 > \beta > 1.0$

(24) region XI,

$x(k) = (\sqrt{b})^k \, (A \cos \theta_5 k + B \sin \theta_5 k)$, unbounded response. $b > 1.0$ (the region XI is inside the parabola $a^2 = 4b$).

where A and B are the constants to be determined from the given initial data.

APPENDIX B

DISCRETE MODEL FOR A DIFFERENTIAL EQUATION

The discrete model of a given differential equation is obtained by application of the central difference operator δ. The central difference operation is defined as follows.

Let x(k) be a real function of integer variable k, the first and second central difference operators δ and δ^2 are defined as

$$\delta x(k) = x(k+\tfrac{1}{2}) - x(k-\tfrac{1}{2})$$

$$\delta^2 x(k) = x(k+1) - 2x(k) + x(k-1) \qquad \text{(B.1)}$$

The two differential equations for which the discrete models are obtained are the Duffing's equation with and without damping. These are indicated below as equations (B.2) and (B.3) respectively. Note that since there may exist a phase difference between x(t) and the impressed force (due to the presence of damping term), for the Duffing equation with damping [5], which is other than 0 and π, the model is assumed to be of the particular form shown in eqn. (B.2). For situation where there is no damping, model in equation (B.3) is adequate.

$$\ddot{x} + \mu c\dot{x} + \alpha x + \mu\gamma x^3 = \mu(F_1 \cos\omega t - F_2 \sin\omega t) \qquad \text{(B.2)}$$

$$\ddot{x} + \alpha x + \mu\gamma x^3 = \mu F \cos\omega t \qquad \text{(B.3)}$$

where $\cdot$ refers to $\frac{d}{dt}$, α, c, γ are constants and μ is small parameter.

The discrete models for equations (B.2), (B.3) are now obtained by application of equation (B.1) as

$$x(k+1) + \lambda\, x(k) + x(k-1) + \mu c\,[x(k+\tfrac{1}{2}) - x(k-\tfrac{1}{2})] + \mu\gamma x^3(k)$$

$$= \mu[F_1 \cos\omega\, k - F_2 \sin\omega\, k] \qquad \text{(B.4)}$$

and

$$x(k+1) + \lambda\, x(k) + x(k-1) + \mu\gamma x^3(k) = \mu F \cos\omega\, k \qquad \text{(B.5)}$$

where

$$\lambda = \alpha - 2.$$

APPENDIX C

POLYNOMIAL APPROXIMATION TO A SATURATION NONLINEARITY

A polynomial approximation for a given saturation nonlinearity is possible for a given restriction on the range of the dependent variable. The saturation nonlinearity and its mathematical representation are given below in Fig. C-1 and in eqn. (C-1) respectively.

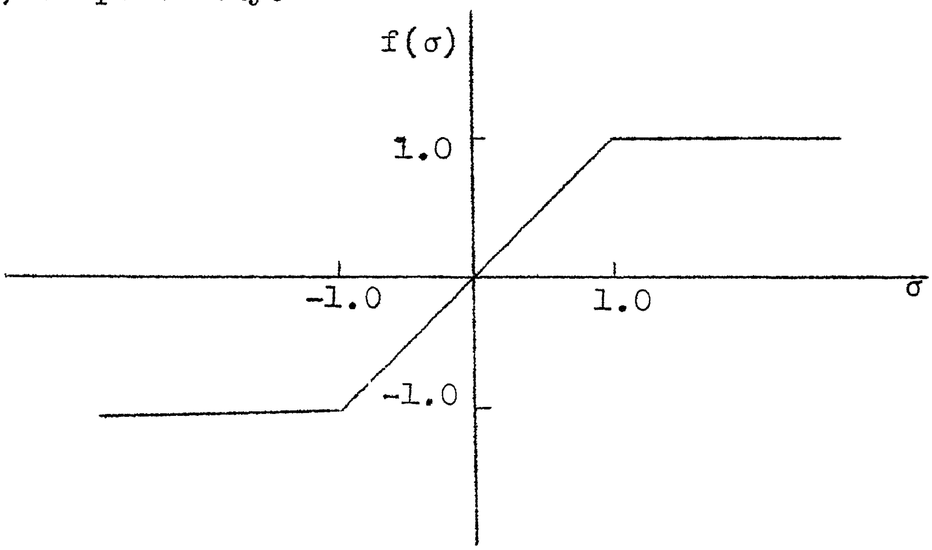


Fig. C-1 Saturation Nonlinearity

$$f(\sigma) = \begin{cases} 1 & \sigma > 1.0 \\ \sigma & |\sigma| \leq 1.0 \\ -1 & \sigma < -1.0 \end{cases} \qquad (C.1)$$

Since the nonlinear characteristic curve shown in Fig. C-1 is confined within first and third quadrants, the polynomial approximation takes the following general form

$$f(\sigma) = a_1\sigma + a_3\sigma^3 + a_5\sigma^5 + \ldots. \qquad (C-2)$$

where a_i are the coefficients to be determined so that the approximation given in eqn. (C-2) is as close as possible to the given saturation nonlinearity.

The coefficients a_i are determined, by assuming a range D for σ, the degree of the polynomial, considering as many points as possible in the range 0 to D and computing using the pseudo inverse approach, as follows.

Assume the degree of the polynomial to be used as m (m odd) and the range of σ be D, that is

$$|\sigma| \leq D$$

Then consider n equal points in the known range 0 to D, namely σ_1, σ_2, σ_3 ..... σ_n. With this construction, the following set of equations are obtained from the eqn. (C-2).

$$f(\sigma_1) = a_1\sigma_1 + a_3\sigma_1^3 + a_5\sigma_1^5 + \ldots\ldots + a_m\sigma_1^m$$

$$f(\sigma_2) = a_1\sigma_2 + a_3\sigma_2^3 + a_5\sigma_2^5 + \ldots\ldots + a_m\sigma_2^m$$

$$f(\sigma_3) = a_1\sigma_3 + a_3\sigma_3^3 + a_5\sigma_3^5 + \ldots\ldots + a_m\sigma_3^m$$

......

......

......

$$f(\sigma_n) = a_1\sigma_n + a_3\sigma_n^3 + a_5\sigma_n^5 + \ldots\ldots + a_m\sigma_n^m \qquad (C-3)$$

Eqn. (C-3) can be rewritten as follows in matrix form :

$$\underline{A}\ \underline{x} = \underline{y} \tag{C-4}$$

where,

$\underline{A}$ is a known rectangular matrix of order $n x (\frac{m+1}{2})$ and is given by

$$\underline{A} = \begin{bmatrix} \sigma_1 & \sigma_1^3 & \sigma_1^5 & \cdots\cdots & \sigma_1^m \\ \sigma_2 & \sigma_2^3 & \sigma_2^5 & \cdots\cdots & \sigma_2^m \\ \sigma_3 & \sigma_3^3 & \sigma_3^5 & \cdots\cdots & \sigma_3^m \\ & \vdots & & \vdots & \\ \sigma_n & \sigma_n^3 & \sigma_n^5 & \cdots\cdots & \sigma_n^m \end{bmatrix}$$

$\underline{x}$ is an $(\frac{m+1}{2})$ unknown vector,

$$\underline{x} = \begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \\ a_m \end{bmatrix}$$

and $\underline{y}$ is a known n vector

$$\underline{y} = \begin{bmatrix} f(\sigma_1) \\ f(\sigma_2) \\ f(\sigma_3) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(\sigma_n) \end{bmatrix}$$

Then multiplying (C-4) by $\underline{A}^T$ (transpose of $\underline{A}$), we have

$$\underline{A}^T \underline{A} \; x = \underline{A}^T \underline{y}$$

$$\underline{x} = (\underline{A}^T \underline{A})^{-1} \; A^T y \qquad \text{(C-5)}$$

eqn. (C-5) gives the values of the coefficients a_i of the polynomial in eqn. (C_2).

For example, with m = 7, D = 3.0, n = 12, the following values for a_i are obtained :

$$a_1 = 1.16253$$
$$a_3 = -0.2850$$
$$a_5 = 0.03643$$
$$a_7 = -0.00167.$$

Thus the polynomial approximation is

$$f(\sigma) = 1.16253 \; \sigma - 0.285 \; \sigma^3 + 0.03643 \; \sigma^5 + 0.00167 \; \sigma^7$$

with

$$|\,\sigma| \leq \quad 3.0 \; .$$

BIO DATA

1. Name : A. KRISHNAN

2. Date of birth : 10.6.1941

3. Educational Qualifications :
   1. B.Sc., 1963, Physics, P.S.G. College, Coimbatore (University of Madras).
   2. B.E., 1966, Electrical Engg., College of Engineering, Guindy, Madras (University of Madras).
   3. M.E., 1974, Electrical and Electronic Engineering (Control Systems), P.S.G. College of Technology, Coimbatore (University of Madras).

Publications

1. 'Determination of Extended Stability region for a class of second order nonlinear discrete systems', J. of Institution of Engineers, India, Vol. 457, 1976. (With Dr. R. Subbayyan and Dr. R. Nagarajan).

2. 'Difference equations in discrete time systems', in Proc. 4th National Systems Conference, held at Coimbatore, 1977 (with Dr. R. Subramanian).

3. 'Limit cycles in digital filters', in Proc. 5th National Systems Conference, held at Ludhiana, 1978. (with Dr. R. Subramanian).

4. 'Nonlinear discrete time systems analysis using multiple scale perturbational technique', to be published in the J. of Sound and Vibration, England, April 1979. (with Dr. R. Subramanian).

5. 'Jump phenomenon in digital filters', Communicated to J. of Electronics and Tele. Comm. Engineers, India (with Dr. R. Subramanian).